Exercises:
Low-rank approximation for nonlinear kinetic problems

Lukas Einkemmer

1. Implement the projector splitting integrator for the 1+1 dimensional Vlasov–Poisson equation. Start by making the simplification that apply to the one-dimensional case. Then discretize in space by introducing and equidistant grid

$$X(t) = \begin{bmatrix} X_1(t,x_1) & \cdots & X_r(t,x_1) \\ \vdots & \ddots & \vdots \\ X_1(t,x_{n_x}) & \cdots & X_r(t,x_{n_x}) \end{bmatrix} \in \mathbb{R}^{n_x \times r}, \qquad V(t) = \begin{bmatrix} V_1(t,v_1) & \cdots & V_r(t,v_1) \\ \vdots & \ddots & \vdots \\ V_1(t,v_{n_v}) & \cdots & V_r(t,v_{n_v}) \end{bmatrix} \in \mathbb{R}^{n_v \times r}$$

   and similarly for K and L. Write the three steps of the projector splitting integrator in matrix form. For example, the K step becomes

$$\partial_t K = -A_{\text{cd}} K (c^1)^T + \text{diag}(E^n) K (c^2)^T,$$

   where A_{cd} is the matrix of the classic second-order finite difference stencil and c^1 and c^2 are the $r \times r$ coefficient matrices introduced in the lecture. In all our computations we approximate the electric field by E^n (the value at the beginning of the time step; this still results in a first order scheme) and use periodic boundary conditions. Couple this with an explicit Runge–Kutta time integrator to implement the algorithm (e.g. in Python or Matlab). Check your implementation by reproducing Landau damping and the two-stream instability.

2. Derive the dynamical low-rank equations of motion for the linear radiative transport equation in diffusive scaling

$$\partial_t f + \frac{1}{\epsilon} v \cdot \nabla_x f = \frac{\sigma^S}{\epsilon^2} \left(\frac{1}{4\pi} \langle f \rangle_v - f \right) - \sigma^A f + G.$$

   The given scattering cross sections σ^S and σ^A do depend on x but not on v. Discretize in space and write the resulting equations in matrix form (similar to exercise 1) and implement the method (e.g. using Python or Matlab).

3. We consider the Boltzmann equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \frac{\nu}{\epsilon} C(f)(x,v), \qquad \nu = \rho T^{1-\omega}$$

   with the BGK collision operator

$$C(f) = M - f, \qquad M(x,v) = \frac{\rho(t,x)}{(2\pi T)^{d/2}} \exp\left(-\frac{1}{2} \frac{(v - u(t,x))^2}{T(t,x)} \right)$$

   and

$$\rho = \int f \, dv, \qquad u = \frac{1}{\rho} \int v f \, dv.$$

Note that $x \in \mathbb{R}^{d_x}$ and $v \in \mathbb{R}^{d_v}$. Show that density ρ, momentum density ρu, and energy density $E = \frac{d_v}{2}\rho T + \frac{1}{2}\rho u^2$ satisfies the compressible Navier–Stokes equations

$$\begin{aligned}
&\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
&\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u + pI) = \varepsilon \nabla_x \cdot (\mu \sigma(u)) + \mathcal{O}(\varepsilon^2), \\
&\partial_t E + \nabla_x \cdot ((E + p)u) = \varepsilon \nabla_x \cdot (\mu \sigma(u) u + \kappa \nabla_x T) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

where

$$\begin{aligned}
\sigma(u) &= \nabla_x u + (\nabla_x u)^T - \frac{2}{d_v}(\nabla_x \cdot u) I \\
p &= \rho T \\
\mu &= T^\omega \\
\kappa &= \frac{d_v + 2}{2}\mu.
\end{aligned}$$

Hint: Start with plugging $f = M + \epsilon f_1$ into the Boltzmann equation to obtain an expression for f_1 in terms of f and M. Then use this to obtain the moments up to $\mathcal{O}(\epsilon)$.

4. Explain the derivation of the Nessyahu–Tadmor scheme as given in the paper `https://doi.org/10.1016/0021-9991(90)90260-8` and implement it for the simple advection equation

$$\partial_t u(t,x) + \partial_x u(t,x) = 0, \qquad x \in [-1, 1], \qquad u(0,x) = \begin{cases} 1 & x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

with periodic boundary conditions. Compare the staggered version of the Nessyahu–Tadmor scheme (using the minmod limiter) with the upwind and the Lax–Friedrichs scheme.