Low-rank approximation for nonlinear kinetic problems
Part 2: Asymptotic preserving dynamical low-rank schemes

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Link to slides: `http://www.einkemmer.net/training.html`

Kinetic equations with a diffusion limit

Transport equation

Transport equation

$$\frac{1}{c}\partial_t I(t,x,\Omega) + \Omega \cdot \nabla_x I(t,x,\Omega) = C(I) - A(I) + G,$$

where $\Omega \in \mathbb{S}^2$ and c is the propagation speed.

Right-hand side depends on the physical problem (Collision, absorption, and source terms).

A **Boltzmann equation** that is of interest in radiation therapy, radiative cooling, neutron transport in nuclear reactors, ...

Main difference with Vlasov equation is **lack of long range interaction.**

Diffusive limit

Linear **radiative transport** equation in **diffusive scaling**

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f = \frac{\sigma^S}{\varepsilon^2}\left(\frac{1}{4\pi}\langle f\rangle_v - f\right) - \sigma^A f + G.$$

A up to **5-dimensional** equation for $f(t,x,v)$ ($x \in \mathbb{R}^3$ and $v \in \mathbb{S}^2$) with complex interplay of transport, collision (relaxation to equilibrium), absorption, and source term.

Collision is the stiffest term in the equation (for small ϵ).

For $\epsilon \to 0$ we obtain the **limit**

$$\partial_t \rho - \nabla_x \cdot \left(\frac{1}{3\sigma^S}\nabla_x \rho\right) = -\sigma^A \rho + G, \qquad \rho = \frac{1}{4\pi}\langle f\rangle_v = \frac{1}{4\pi}\int_{\mathbb{S}^2} f \, \mathrm{d}v.$$

This **solution has rank 1** with $f(t,x,v) = \rho(t,x)$.

Diffusive limit

We start with a Hilbert expansion/Chapman–Enskog theory

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots$$

Plugging into the radiative transport equation (neglecting absorption and source terms)

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f = \frac{\sigma}{\varepsilon^2} \left(\frac{1}{4\pi} \langle f \rangle_v - f \right)$$

and collecting terms of the same order in ϵ.

$\mathcal{O}(1/\epsilon^2)$: $f_0 = \langle f_0 \rangle_v / (4\pi)$ and thus f_0 does not depend on v.

- For $\epsilon \to 0$ we have $f = \rho$.

$\mathcal{O}(1/\epsilon)$:

$$v \cdot \nabla_x f_0 = \sigma \left(\frac{1}{4\pi} \langle f_1 \rangle_v - f_1 \right)$$

Diffusive limit

$\mathcal{O}(1)$:

$$\partial_t f_0 + v \cdot \nabla_x f_1 = \sigma \left(\frac{1}{4\pi} \langle f_2 \rangle_v - f_2 \right) \quad \Longrightarrow \quad \partial_t \langle f_0 \rangle_v + \nabla_x \cdot \langle v f_1 \rangle_v = 0$$

From the $1/\epsilon$ term we get by multiplying with v and integrating

$$v \cdot \nabla_x f_0 = \sigma \left(\frac{1}{4\pi} \langle f_1 \rangle_v - f_1 \right) \quad \Longrightarrow \quad \langle v f_1 \rangle_v = -\frac{1}{\sigma} \langle v \nabla_x \cdot (v f_0) \rangle_v = -\frac{1}{3\sigma} \nabla_x f_0$$

Putting everything together and using $\rho = \langle f_0 \rangle_v / 4\pi + \mathcal{O}(\epsilon)$ we get the diffusion equation

$$\partial_t \rho = \nabla_x \cdot \left(\frac{1}{3\sigma} \nabla_x \rho \right) + \mathcal{O}(\epsilon).$$

- We have to get up to *third order* in ϵ to get the dynamics of f_0.
- The collision operator determines how the equilibrium looks like.
- The transport determines the dynamics of the equilibrium.

Interest in the limit

For ϵ small we obtain information on the **low-rank structure of the solution**

$$f(t,x,v) = f_0(t,x) + \epsilon \left(\frac{1}{4\pi} \langle f_1 \rangle_v - \frac{v}{\sigma} \cdot \nabla_x f_0 \right) + \mathcal{O}(\epsilon^2)$$

Up to $\mathcal{O}(\epsilon)$ the solution is **at most rank 4.**

This is one of the **few situations where we do know how the low-rank structure of the solution looks like.**

Sidenote: If we assume enough smoothness. Then,

$$f(x,v) \approx \sum_{k=1}^{n} \sum_{m=1}^{n} \hat{f}_{km} \mathrm{e}^{ikx} \mathrm{e}^{imv}$$

gives a low-rank approximation with moderate rank. Usually not a valid assumption for hyperbolic problems.

Error analysis for DLR in the diffusion limit

Dynamical low-rank integrator

We again use the **projector splitting integrator**.

L step in the continuous setting

$$\partial_t L_i = -\frac{1}{\epsilon}\sum_{j=1}^{r} v \cdot \langle X_i^n, \nabla X_j^n \rangle_x L_j + \frac{1}{\epsilon^2}\sum_{j=1}^{r} \langle X_i^n \sigma X_j^n \rangle_x \left(\langle L_j \rangle_v - L_j \right).$$

For $\epsilon \to 0$ we must have $L_j = \langle L_j \rangle_v$, **L looses dependence on v**.

Discretization

Discretization with $\mathrm{L}^n = [L_1(v_\cdot), L_2(v_\cdot), \ldots, L_r(v_\cdot)] \in \mathbb{R}^{n_v \times r}$

$$\frac{\mathrm{L}^\star - \mathrm{L}^n}{\Delta t} + \frac{1}{\epsilon}\sum_{k=1}^{d} \mathrm{A}^n_{\partial_k} \mathrm{L}^\star \Pi_{v_k} = \frac{\mathrm{A}^n_\sigma}{\epsilon^2} \mathrm{L}^\star \mathrm{C},$$

$$\text{QR decomposition: } \mathrm{L}^\star = \mathrm{V}^\star (\mathrm{S}^\star)^\top,$$

where

$$\mathrm{A}^n_{\partial_k} = (\mathrm{X}^n)^\top \mathrm{D}_k \mathrm{X}^n, \quad \mathrm{A}^n_\sigma = \mathrm{X}^\top \Sigma \mathrm{X}, \quad \Pi_{v_k} = \mathrm{diag}(v_k), \quad \mathrm{C} = \frac{1}{n_v} e e^\top - \mathrm{I}$$

with

$$e = (1, 1, \ldots, 1)^\top, \qquad \Sigma = \mathrm{diag}(\sigma), \qquad \mathrm{D}_k \approx \partial_{x_k}.$$

We have used an implicit Euler step here.

Implicit Euler integrator

Input: X^n, S^n and V^n.
Algorithm:

- ▶ Construct $A^n_{\partial_k}, A^n_\sigma$; Compute $L^n = V^n(S^n)^\top$;
- – Integrate L step with step size Δt to get $L^\star$;
- ▶ Perform QR decomposition to obtain $S^\star, V^{n+1}$; Construct $\Xi^{n+1}_{v_k}$ and Γ^{n+1};
- – Integrate S step with step size Δt to get $S^{\star\star}$ with Δt;
- ▶ Construct $K^{\star\star} = X^n S^{\star\star}$;
- – Integrate K step with step size Δt to get K^{n+1} ;
- ▶ Perform QR decomposition to obtain X^{n+1}, S^{n+1};

Output: $X^{n+1}, S^{n+1}, and V^{n+1}$;

We do the **L step first because it drives our system into equilibrium**.

Error analyis

Theorem (Implicit Euler integrator)

Under technical assumptions there is a constant C independent of ϵ so that:

$$\|f^{n+1} - \rho_0^{n+1} e^\top\|_2 \leq C\epsilon\,,$$

where ρ_0^{n+1} solves

$$\frac{\rho_0^{n+1} - \rho_0^{\star\star}}{\Delta t} = \frac{1}{d}\sum_{k=1}^{d} \mathrm{D}_k \left(\Sigma^{-1}\mathrm{D}_k\rho_0^{n+1}\right) \tag{1}$$

and $\|\rho_0^{\star\star} - \rho_0^{n}\|_2 = O\left(\frac{(\Delta t)^2}{(\Delta x)^4}\right)$.

The good: Captures the correct low-rank structure and correct discretization of limit equation.

The bad: There is a strong constraint on $\Delta t \ll (\Delta x)^2$ that makes the scheme unfeasible in practice.

Proof sketch

All quantities of interested are expanded using the ansatz

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots .$$

We have three steps in the projector splitting integrator (equations for L, S, and K).

L step: Asymptotic expansion

$$\begin{cases} O(1/\epsilon^2): & \mathsf{A}_\sigma^n \mathsf{L}_0^\star \mathsf{C} = 0\,, \\ O(1/\epsilon): & \sum_{k=1}^d \mathsf{A}_{\partial_k}^n \mathsf{L}_0^\star \Pi_{v_k} = \mathsf{A}_\sigma^n \mathsf{L}_1^\star \mathsf{C}\,, \\ O(1): & \frac{\mathsf{L}_0^\star - \mathsf{L}_0^n}{\Delta t} + \sum_{k=1}^d \mathsf{A}_{\partial_k}^n \mathsf{L}_1^\star \Pi_{v_k} = \mathsf{A}_\sigma^n \mathsf{L}_2^\star \mathsf{C}\,. \end{cases}$$

This gives the representation of the equilibrium

$$\mathsf{L}_0^\star = l_0^\star e_{\mathrm{n}}^\top .$$

Proof sketch

K step: Asymptotic expansion yields (with $\alpha^{n+1} = (\mathsf{V}^{n+1})^\top \mathrm{e_n}$)

$$\frac{\mathsf{K}_0^{n+1}\alpha^{n+1} - \mathsf{K}_0^{\star\star}\alpha^{n+1}}{\Delta t} - \frac{1}{d}\sum_{k=1}^{d} \mathsf{D}_k \Sigma^{-1} \mathsf{D}_k \mathsf{K}_0^{n+1}\alpha^{n+1} = 0$$

and thus

$$\frac{\rho_0^{n+1} - \rho_0^{\star\star}}{\Delta t} - \frac{1}{d}\sum_{k=1}^{d} \mathsf{D}_k \left(\Sigma^{-1}\mathsf{D}_k \rho_0^{n+1}\right) = 0.$$

S step together with the L step is a complication (ideally $\rho_0^{\star\star} = \rho_0^n$). But

$$\rho_0^{\star\star} = \left(\mathsf{I} + \frac{\Delta t}{d}\mathcal{L}\right)^{-1}\left(\mathsf{I} - \frac{\Delta t}{d}\mathcal{L}\right)^{-1}\rho_0^n = \left(\mathsf{I} - \frac{(\Delta t)^2}{d^2}\mathcal{L}^2\right)^{-1}\rho_0^n,$$

where $\mathcal{L} = \sum_{k=1}^{d} \mathsf{X}^n \mathsf{A}_{\partial_k}^n (\mathsf{A}_\sigma^n)^{-1} \mathsf{A}_{\partial_k}^n (\mathsf{X}^n)^\top$.

CNIE integrator

Input: X^n, S^n and V^n.
Algorithm:

- Construct $A^n_{\partial_k}, A^n_{\sigma}$; Compute $L^n = V^n(S^n)^\top$;
  - Integrate L step with step size Δt to get $L^\star$ using Crank-Nicolson;
- Perform QR decomposition to obtain $S^\star, V^{n+1}$; Construct $\Xi^{n+1}_{v_k}$ and Γ^{n+1};
  - Integrate S step with step size Δt to get $S^{\star\star}$ with Δt using Crank-Nicolson;
- Construct $K^{\star\star} = X^n S^{\star\star}$;
  - Integrate K step with step size Δt to get K^{n+1};
- Perform QR decomposition to obtain X^{n+1}, S^{n+1};

Output: $X^{n+1}, S^{n+1}, and V^{n+1}$;

Error analysis

Crank-Nicolson for the L step:

$$\frac{\mathrm{L}^\star - \mathrm{L}^n}{\Delta t} + \frac{1}{\epsilon}\sum_{k=1}^{d} \mathrm{A}^n_{\partial_k}\left(\frac{\mathrm{L}^\star + \mathrm{L}^n}{2}\right)\Pi_{v_k} = \frac{\mathrm{A}^n_\sigma}{\epsilon^2}\frac{\mathrm{L}^\star + \mathrm{L}^n}{2}\mathrm{C}.$$

Theorem (CNIE integrator)

Assuming that $f^n = \rho^n_0 e^\top + \mathcal{O}(\epsilon)$ *we get*

$$\|\rho^{\star\star} - \rho^n\|_2 = O(\epsilon)$$

and

$$\frac{\rho_0^{n+1} - \rho_0^{\star\star}}{\Delta t} - \frac{1}{d}\sum_{k=1}^{d} \mathrm{D}_k\left(\Sigma^{-1}\mathrm{D}_k\rho_0^{n+1}\right) = 0.$$

Further, we have $f^{n+1} = \rho_0^{n+1} e^\top + \mathcal{O}(\epsilon)$.

The low-rank limit is obtained independent of the time step size.

Proof sketch

Much of the proof proceeds analogous to the implicit Euler case.

But, because of the Crank–Nicolson scheme in the L and S step we now have

$$\rho_0^{\star} = \left(\mathrm{I} - \frac{\Delta t}{2d}\mathcal{L}\right)^{-1}\left(\mathrm{I} + \frac{\Delta t}{2d}\mathcal{L}\right)\rho_0^n,$$
$$\rho_0^{\star\star} = \left(\mathrm{I} + \frac{\Delta t}{2d}\mathcal{L}\right)^{-1}\left(\mathrm{I} - \frac{\Delta t}{2d}\mathcal{L}\right)\rho_0^{\star},$$

where as before $\mathcal{L} = \sum_{k=1}^{d} \mathrm{X}^n \mathrm{A}^n_{\partial_k}(\mathrm{A}^n_{\sigma})^{-1}\mathrm{A}^n_{\partial_k}(\mathrm{X}^n)^{\top}$.

- Note that since the S step (second line) is backward in time the signs match up.

Combining these two equations we get $\rho_0^{\star\star} = \rho_0^n$.

Final thoughts

Hilbert expansion as discussed here is a formal method.

Proofs can be made rigorous by the techniques in
[C. Bardos, R. Santos, and R. Sentis, Trans. Amer. Math. Soc., 284 (1984)]

Efficient asymptotic preserving schemes

Efficiency

The previously discussed numerical method are **fully implicit.**

Requires the solution of a **large linear systems** in every time step.

- Very expensive!

Macro-micro decomposition

Macro-micro decomposition

$$f(t,x,v) = \rho(t,x) + \varepsilon g(t,x,v).$$

Evolution equation for ρ and g

$$\partial_t \rho + \frac{1}{4\pi}\nabla_x \cdot \langle v g\rangle_v = -\sigma^A \rho + G$$

$$\partial_t g + \frac{1}{\varepsilon}\left(I - \frac{1}{4\pi}\langle\ \rangle_v\right)(v\cdot\nabla_x g) + \frac{1}{\varepsilon^2} v\cdot\nabla_x \rho = -\frac{\sigma^S}{\varepsilon^2} g - \sigma^A g.$$

For $\epsilon \to 0$ we have

$$g = -\frac{1}{\sigma^S} v\cdot\nabla_x \rho,$$

which is at most **rank d**.

We now approximate g using a low-rank representation.

M. Lemou, L. Mieussens, SIAM J. Sci. Comput., 31:334–368, 2008.

Evolution equation

Evolution equation for *K*

$$\partial_t K_j = \langle V_j, \mathrm{RHS} \rangle_v$$
$$= -\frac{1}{\varepsilon}\sum_{l=1}^{r}\left(\langle vV_j V_l\rangle_v - \frac{1}{4\pi}\langle V_j\rangle_v \langle vV_l\rangle_v\right)\cdot\nabla_x K_l - \frac{1}{\varepsilon^2}\langle vV_j\rangle_v \cdot \nabla_x \rho - \left(\frac{\sigma^S}{\varepsilon^2} + \sigma^A\right) K_j.$$

First order **IMEX** discretization

$$\frac{K_j^{n+1} - K_j^n}{\Delta t} = -\frac{1}{\varepsilon}\sum_{l=1}^{r}\left(\langle vV_j^n V_l^n\rangle_v - \frac{1}{4\pi}\langle V_j^n\rangle_v \langle vV_l^n\rangle_v\right)\cdot\nabla_x K_l^n$$
$$-\frac{1}{\varepsilon^2}\left(\langle vV_j^n\rangle_v \cdot \nabla_x \rho^n + \sigma^S K_j^{n+1}\right) - \sigma^A K_j^n.$$

is **AP and explicit**.

Second order

We have the macro-micro equations

$$\partial_t \rho = F(\rho, g), \qquad \partial_t g = G(g, \rho).$$

First order can be easily obtained by substituting the following set of equations

$$\partial_t \rho = F(\rho, g^n), \qquad \partial_t g = G(g, \rho^n).$$

For second order we can exploit symmetry. If $g^{n+1/2}$ and $\rho^{n+1/2}$ are first order approximations to $g(t^n + \Delta t/2)$ and $\rho(t^n + \Delta t/2)$ then

$$\partial_t \rho = F(\rho, g^{n+1/2}), \qquad \partial_t g = G(g, \rho^{n+1/2}).$$

is second order accurate (assuming the equations are solved up to second order).

Almost symmetric splitting

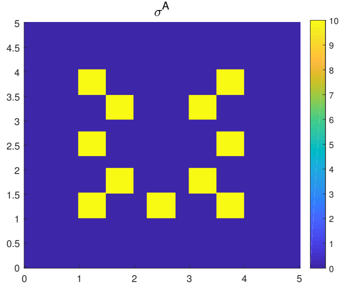
The following method is **second order accurate and AP**

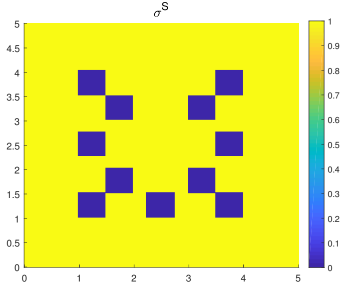
- Compute $\rho^{n+1/2}$ from ρ^n using g^n (first order is sufficient).
- Solve $\partial_t g = G(g, \rho^{n+1/2})$ using projector splitting and IMEX2.
- Compute ρ^{n+1} from ρ^n using $g(t^{n+1/2})$ (using a second order scheme).

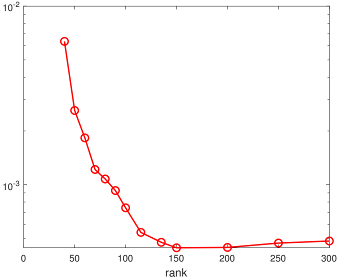
Can be generalized to higher order (almost-symmetric splitting).

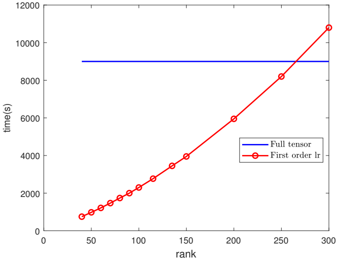
L.E., A. Ostermann, Comput. Appl. Math., 271, 2014.
L.E., A. Ostermann, Comput. Math. Appl., 67(12), 2014.

Two-material test problem

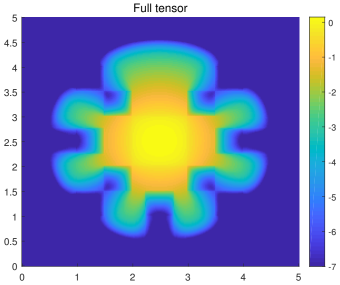


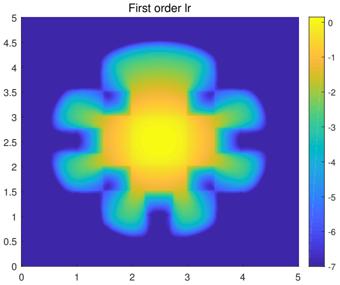


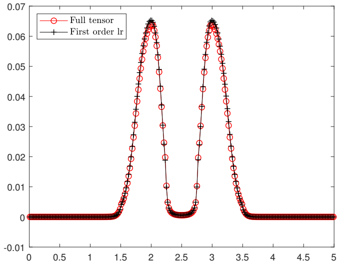


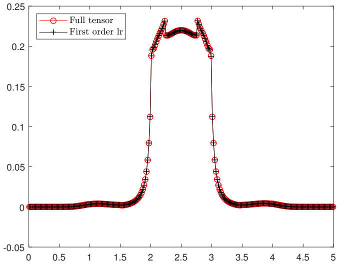


Two-material test problem









Kinetic equations with a fluid limit

Fluid limit

Collisional kinetic equation with a fluid limit

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \frac{1}{\epsilon}(M(f) - f) \quad \xrightarrow{\epsilon \to 0} \quad \boxed{\textbf{Euler equations}}$$

with

$$M = \frac{\rho}{(2\pi)^{d_v/2}} \exp\left(-\frac{|v-u|^2}{2}\right), \qquad \rho = \int f \, \mathrm{d}v, \qquad u = \frac{1}{\rho}\int vf \, \mathrm{d}v.$$

Chapman–Enskog theory: $f = M(1 + \epsilon f_1 + \epsilon^2 f_2)$

Density and momentum satisfy the (isothermal) **Navier–Stokes equation**

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$
$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \rho = \nabla \cdot \left[\mu\left(\nabla u + (\nabla u)^{\mathrm{T}}\right) + \lambda(\nabla \cdot u) I\right].$$

C. Bardos, F. Golse, and D. Levermore. J. Stat. Phys, 63(1-2), 1991.
C. Bardos, F. Golse, and C.D. Levermore. Commun. Pure Appl. Math, 46(5), 1993.

Low-rank

For $\epsilon \to 0$ we have

$$f = M(\rho, u) = \frac{\rho}{(2\pi)^{d_v/2}} \exp\left(-\frac{|v-u|^2}{2}\right).$$

Dynamics completely determined by the moments ρ and u (fluid regime).

But f is **not low-rank**.

Interlude: Lattice Boltzmann methods

At least two strategies to solve fluid problems.

1. **Directly discretize the Navier–Stokes equations.**
2. **Discretize the Boltzmann equation**

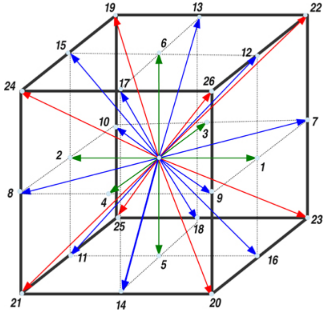
$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\epsilon}(M(f) - f)$$

with a very coarse velocity discretization

$$f_j(t,x) \approx f_j(t,x,e_j).$$

Then reconstruct the quantities of interest

$$\rho(t,x) = \int f(t,x,v)\,\mathrm{d}v \approx \sum_j \omega_j f_j(t,x).$$



Low-rank vs Lattice Boltzmann

Why throw away all the information about velocity space?

- Even in a fluid problem this is still important.

Apply the low-rank algorithm with initial value

$$f(0,x,v) = \frac{\rho(0,x)}{(2\pi)^{d_v/2}} \exp\left(-\tfrac{1}{2}(v - u(0,x))^2\right).$$

Advantages

- For weakly compressible flow we can use (rank 10 in 3d)
$$\frac{\rho}{(2\pi)^{d/2}} \exp\left(-\frac{v^2}{2}\right)\left(1 + v \cdot u + \frac{(v \cdot u)^2}{2} - \frac{u^2}{2}\right) + \mathcal{O}\left(u^3\right).$$
- Spectral methods can be incorporated easily.
- Straightforward to capture some kinetic effects.
- Additional cost is small.

Low-rank approximation

Evolution equation for K

$$\partial_t K_j(t,x) = -\sum_l c^1_{jl} \cdot \nabla_x K_l(t,x) + \frac{1}{\epsilon}\Big(K_j(t,x) - c^3_j(K)(t,x)\rho(K)(t,x)\Big)$$

In the limit $\epsilon \to 0$ we have

$$K_j - c^3_j(K)(x)\rho(K)(x) = 0$$

which can be written as

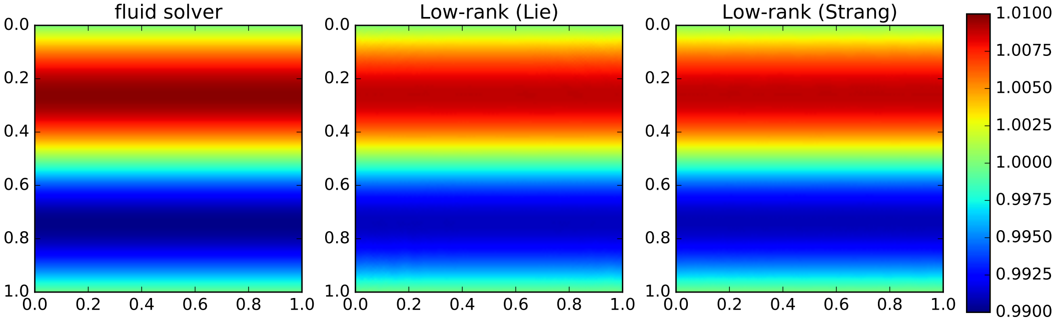
$$K_j = \frac{\rho(K)}{(2\pi)^{d/2}} \int V_j(v) \exp\left(-\tfrac{1}{2}(v - u(K))^2\right) \mathrm{d}v.$$

This is simply the **projection of the Maxwell–Boltzmann distribution onto the space spanned by the $\mathbf{V_j}$**.

Sound waves

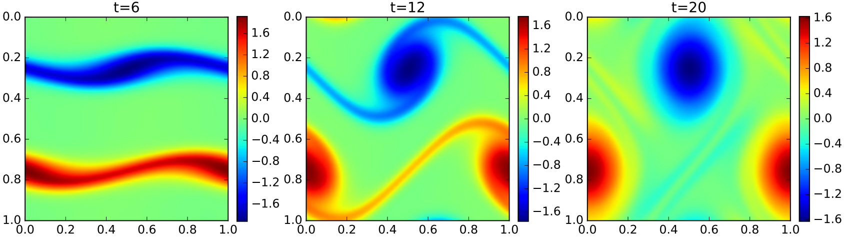
Propagation of sound waves

- Fluid solver with $\tau = 7 \cdot 10^{-3}$ (CFL number of 0.9).
- Lie splitting with $\tau = 0.1$ and Strang splitting with $\tau = 0.2$.

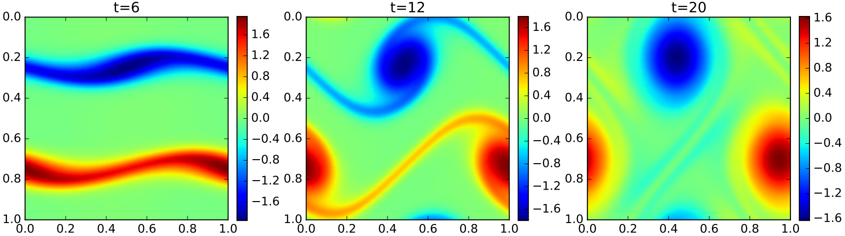


Shear flow

Fluid solver



Low-rank (Strang/FFT, $n_v = 16$)



The compressible case

Low-rank structure

Is there a low-rank structure in the compressible case?

- Note that equilibrium is uniquely defined by its moments.

In the macro-micro decomposition we have $f = M + \varepsilon f_1$. Plugging this into

$$\partial_t f + v \cdot \nabla_x f = \frac{\nu}{\varepsilon}(M - f).$$

and using $(M - f)/\varepsilon = f_1$ we get

$$f_1 = -\frac{1}{\nu}(\partial_t f + v \cdot \nabla_x f) = -\frac{1}{\nu}(\partial_t M + v \cdot \nabla_x M) + \mathcal{O}(\varepsilon).$$

Almost a low-rank structure

$$\begin{aligned} &\frac{1}{M}(\partial_t M + v \cdot \nabla_x M) \\ &\quad = \frac{1}{\rho}(\partial_t \rho + v \cdot \nabla_x \rho) + (v - u) \cdot (\partial_t u + v \cdot \nabla_x u). \end{aligned}$$

Decomposition

The micro-macro decomposition fails because $f_1 = M(\text{low-rank})$.

We will instead use the **multiplicate decompositon** $f = Mg$.

Then

$$g = 1 - \frac{\varepsilon}{\nu}\left[\left((v-u)\otimes(v-u) - \frac{|v-u|^2}{d_v} I\right) : \nabla_x u\right] + \mathcal{O}(\varepsilon^2),$$

where $A : B = \sum_{ij} A_{ij} B_{ij}$.

g is **low-rank up to at least** $\mathcal{O}(\epsilon^2)$.

Multiplicative micro-macro decompositin

We have

$$f = Mg, \qquad g = \sum_{ij} X_i S_{ij} V_j.$$

Evolution of the moments to obtain M

$$\partial_t U + \nabla_x \cdot \langle v\phi M g\rangle_v = 0, \quad U = (\rho, \rho u)^T, \quad \phi(v) = (1, v)^T.$$

Treated as any other conservation law.

Evolution of g

$$\partial_t g = -v \cdot \nabla_x g - \frac{1}{M}(\partial_t M + v \cdot \nabla_x M)g + \frac{\nu}{\varepsilon}(1 - g).$$

Treated by dynamical low-rank.

Challenges

For **DLR** this moves the problem into the **coefficients**. E.g.

$$\langle v V_j^n M^n \rangle_v = \frac{\rho^n(x)}{(2\pi)^{d_v/2}} \left\langle v V_j^n(v) \exp\left(-\frac{|v-u^n(x)|^2}{2}\right)\right\rangle_v .$$

$$\langle (v \otimes v) V_j^n M^n \rangle_v = \frac{\rho^n(x)}{(2\pi)^{d_v/2}} \left\langle (v \otimes v) V_j^n(v) \exp\left(-\frac{|v-u^n(x)|^2}{2}\right)\right\rangle_v$$

Can be treated by **fast convolution algorithms**

$$g_j^1 = (v \mapsto v V_j^n) * (v \mapsto \exp(-v^2/2)), \qquad \text{evaluated at} \qquad u^n(x).$$

$$g_j^2 = (v \mapsto (v \otimes v) V_j^n) * (v \mapsto \exp(-v^2/2)), \qquad \text{evaluated at} \qquad u^n(x).$$

Fast convolution

To compute the convolution $h_1 * h_2$ of $h_1(v)$ and $h_2(v)$ we proceed as follows.

Step 1: Compute $\hat{h}_1$ and $\hat{h}_2$ by using a FFT.

- Cost: $\mathcal{O}(n^{d_v} \log n^{d_v})$.

Step 2: Compute $g = \mathcal{F}^{-1}(\hat{h}_1 \hat{h}_2)$.

- Cost: $\mathcal{O}(n^{d_v} \log n^{d_v})$.

Step 3: Interpolate g (e.g. using cubic spline interpolation) and evaluate $g(u(x_i))$ for each grid point x_i

- Cost: $\mathcal{O}(n^{d_x})$.

Shock waves

In the Euler limit ($\epsilon \to 0$) **shock waves** are known to develop.

- Discontinuous solutions or for $\epsilon > 0$ sharp gradients in the solution.

It is well known that standard methods do not work in this case.

This is a significant complication compared to the diffusion limit.

K equation

Discretizing the K equation

$$\partial_t K_j = -\sum_l (\nabla_x K_l) \cdot \langle v V_j V_l \rangle_v - \sum_l K_l \langle V_j V_l M \rangle_v + \frac{\rho}{\varepsilon} \left(\langle V_j \rangle_v - K_j \right)$$

using a first order IMEX scheme

$$K_j^{n+1} = \frac{1}{1 + \Delta t \rho^n / \varepsilon} K_j^n - \frac{\Delta t}{1 + \Delta t \rho^n / \varepsilon} \left[\sum_l c_{jl}^1 \cdot (\nabla_x K_l^n) + \sum_l c_{jl}^2 K_l^n \right] + \frac{\Delta t \rho^n}{\varepsilon + \Delta t \rho^n} \langle V_j \rangle_v .$$

In 2D $c_{jl}^1 = [c_{jl}^{1;1} \; c_{jl}^{1;2}]^{\mathrm{T}}$ and the matrices are symmetric. Thus, there exist orthogonal matrices T^m such that $\sum_{jl} T_{ij}^m c_{jl}^{1;m} T_{kl}^m = \lambda_i^m \delta_{ik}$.

K equation

Using $\hat{K}_i^n = \sum_j T_{ij}^1 K_j^n$ we get

$$\hat{K}_i^{n+1} = \frac{1}{1+\Delta t \rho^n/\varepsilon}\hat{K}_i^n - \frac{\Delta t}{1+\Delta t \rho^n/\varepsilon}\left[\lambda_i^1 \partial_x \hat{K}_i^n + \sum_{jl} T_{ij}^1 c_{jl}^{1;2} \partial_y K_l^n + \sum_{lj} T_{ij}^1 c_{jl}^2 K_l^n\right] + \ldots$$

Direction of the flow is now obvious.

Replace $\lambda_i^1 \partial_x \hat{K}_i^n$ by an appropriate discrete approximation, which we denote by $\delta_x(K_i^n, \lambda_i^1)$.

$$K_j^{n+1} = \frac{1}{1+\Delta t \rho^n/\varepsilon} K_j^n - \frac{\Delta t}{1+\Delta t \rho^n/\varepsilon}\left[\sum_i T_{ij}^1 \delta_x(\hat{K}_i^n, \lambda_i^1) + \sum_l c_{jl}^{1;2} \partial_y K_l^n + \sum_l c_{jl}^2 K_l^n\right] + \ldots$$

Examples: upwinding, Lax–Wendroff flux with van Leer limiter, ...

Moment equation

For the moments $U = (\rho, \rho u)^T$ we have

$$\partial_t U + \nabla_x \cdot \langle v \phi M g \rangle_v = 0, \qquad \phi(v) = (1, v)^T.$$

This is in the form of a conservation flow, **but the flux depends on g**.

- Classic methods for conservation flow solve a Riemann problem.
- This is difficult to do here.

Central schemes

We consider the one-dimensional conservation law

$$\partial_t U(t,x) + \partial_x F(U(t,x)) = 0.$$

Lax-Friedrichs method

$$U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(F(U_{j+1}^n) - F(U_{j-1}^n)\right)$$

is stable under the usual CFL condition.

Simplest example of a central scheme.

- Stable scheme that does not require information on the direction of the flow.

Nessyahu–Tadmor

The **Nessyahu–Tadmor** scheme is a second-order (away from discontinuities) central scheme in predictor-corrector form on a staggered grid

$$U_j^\star = U_j^n - \frac{\Delta t}{2\Delta x} F_j',$$
$$U_{j+1/2}^{n+1} = \frac{1}{2}(U_j^n + U_{j+1}^n) + \frac{1}{8}(U_j' - U_{j+1}') - \frac{\Delta t}{\Delta x}(F(U_{j+1}^\star) - F(U_j^\star)).$$

The choice of F_j' and U_j' is free as long as

- $F_j'/\Delta x = \partial_x F(U(t^n, x_j)) + \mathcal{O}(\Delta x)$
- $U_j'/\Delta x = \partial_x U(t^n, x_j) + \mathcal{O}(\Delta x)$.

To obtain non-oscillatory solutions that preserve sharp gradient we can choose

$$U_j' = \mathrm{MM}(U_{j+1}^n - U_j^n, U_j^n - U_{j-1}^n), \qquad F_j' = \mathrm{MM}(F(U_{j+1}^n) - F(U_j^n), F(U_j^n) - F(U_{j-1}^n))$$

with MM the usual minmod limiter.

Generalization to 2D

$$\begin{aligned}
U_{ij}^{\star} &= U_{ij}^{n} - \frac{\Delta t}{2\Delta x} F_{ij}^{\prime x} - \frac{\Delta t}{2\Delta y} G_{ij}^{\prime y}, \\
U_{i+1/2,j+1/2}^{n+1} &= \frac{1}{4}\left(U_{ij}^{n} + U_{i+1,j}^{n} + U_{i,j+1}^{n} + U_{i+1,j+1}^{n}\right) \\
&\quad + \frac{1}{16}\left(U_{ij}^{\prime x} - U_{i+1,j}^{\prime x} + U_{i,j+1}^{\prime x} - U_{i+1,j+1}^{\prime x}\right) \\
&\quad + \frac{1}{16}\left(U_{ij}^{\prime y} - U_{i,j+1}^{\prime y} + U_{i+1,j}^{\prime y} - U_{i+1,j+1}^{\prime y}\right) \\
&\quad + \frac{\Delta t}{\Delta x}\left(F(U_{i+1,j}^{\star}) - F(U_{ij}^{\star})\right) + \frac{\Delta t}{\Delta y}\left(F(U_{i,j+1}^{\star}) - F(U_{ij}^{\star})\right),
\end{aligned}$$

where

$$\begin{aligned}
U_{ij}^{\prime x} &= \mathrm{MM}(U_{i+1,j}^{n} - U_{i,j}^{n}, U_{ij}^{n} - U_{i-1,j}^{n}), \qquad U_{ij}^{\prime y} = \mathrm{MM}(U_{i,j+1}^{n} - U_{ij}^{n}, U_{ij}^{n} - U_{i,j-1}^{n}), \\
F_{ij}^{\prime x} &= \mathrm{MM}(F(U_{i+1,j}^{n}) - F(U_{i,j}^{n}), F(U_{ij}^{n}) - F(U_{i-1,j}^{n})), \\
F_{ij}^{\prime y} &= \mathrm{MM}(F(U_{i,j+1}^{n}) - F(U_{ij}^{n}), F(U_{ij}^{n}) - F(U_{i,j-1}^{n})).
\end{aligned}$$

Dynamical low-rank approximation

In our case we have

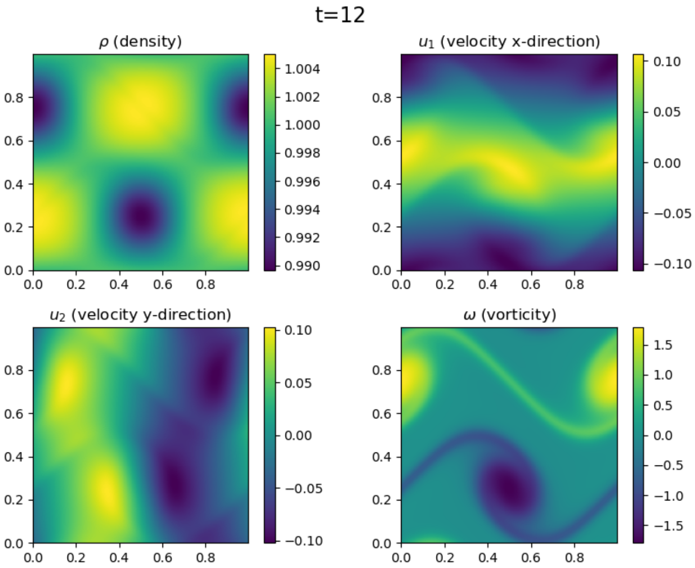
$$U = \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \qquad F(U) = \begin{bmatrix} \sum_{ij} X_i^n S_{ij}^n \langle v V_j^n M^n \rangle_v \\ \sum_{ij} X_i^n S_{ij}^n \langle (v \otimes v) V_j^n M^n \rangle_v \end{bmatrix}.$$

To compute the flux we use the fast convolution algorithm.

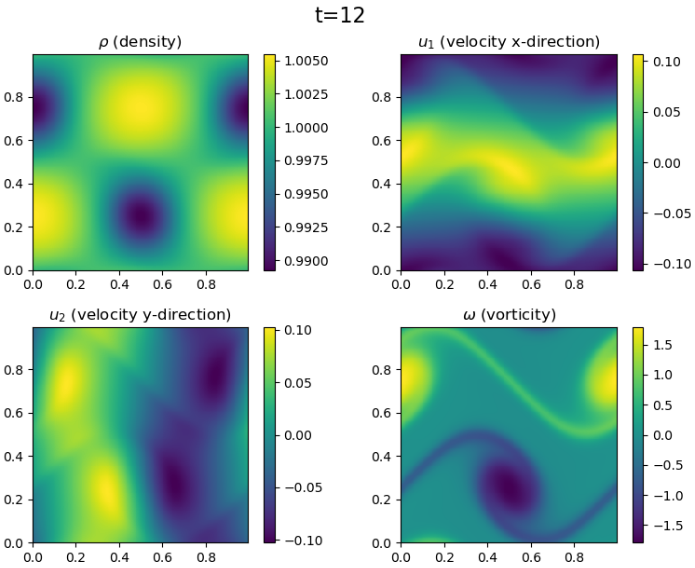
We found that the **staggered variant is more accurate**

1. Apply the Nessyahu–Tadmor scheme with time step size $\Delta t/2$ and initial value U_{ij}^n to obtain $U_{i+1/2,j+1/2}^{n+1/2}$.
2. Obtain X^n at the half grid points $(i+1/2, j+1/2)$ by computing averages between neighboring grid points.
3. Apply the Nessyahu–Tadmor scheme with time step size $\Delta t/2$ and initial value $U_{i+1/2,j+1/2}^{n+1/2}$ and the X^n at the half grid points to obtain $U_{i,j}^{n+1}$.

Shear flow (fluid solver)

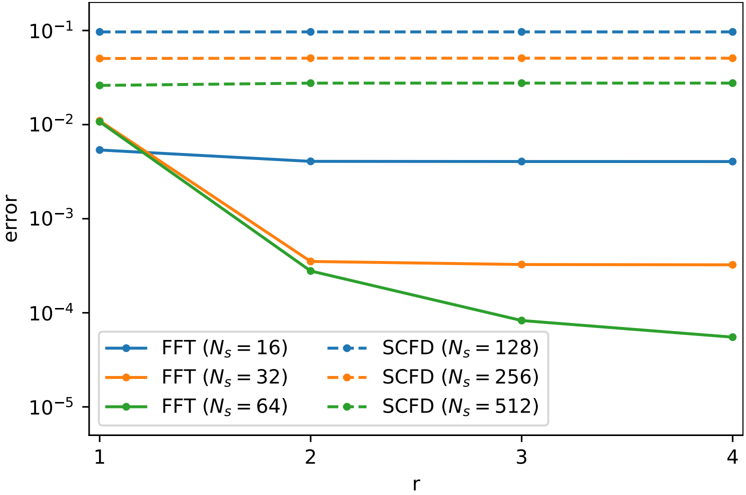


Shear flow (dynamical low-rank)

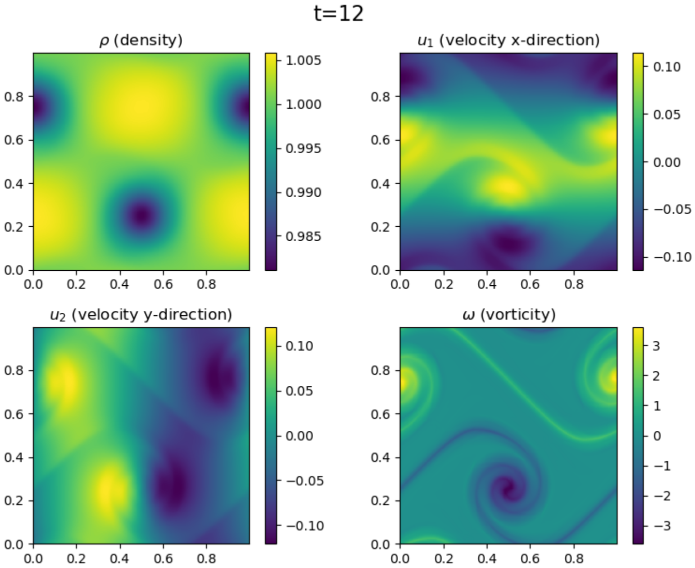


Error

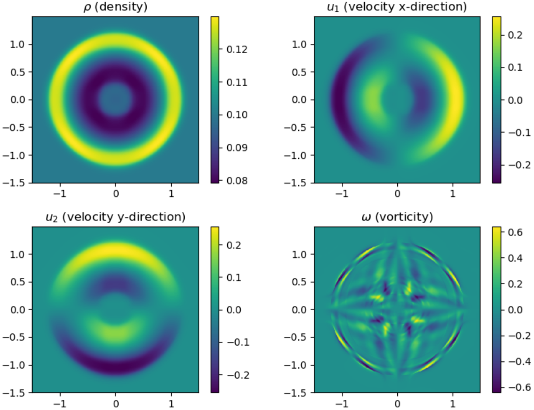
Error in the moments for different rank r and space discretizations



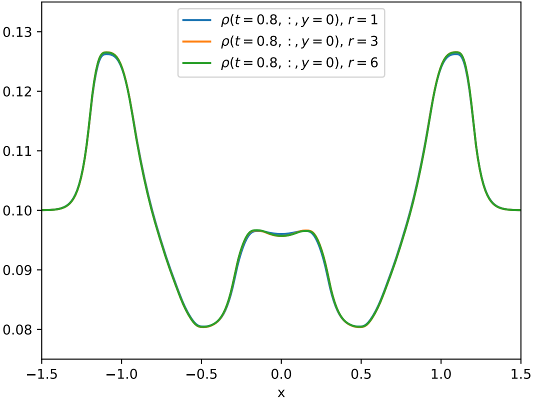
Shear flow with large Reynolds number



Explosion



ρ (density)
u_1 (velocity x-direction)
u_2 (velocity y-direction)
ω (vorticity)



$\rho(t = 0.8, :, y = 0)$, $r = 1$
$\rho(t = 0.8, :, y = 0)$, $r = 3$
$\rho(t = 0.8, :, y = 0)$, $r = 6$
x

Literature

Literature

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Literature

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[L.E., J. Hu, L. Ying. arXiv:2101.07104]

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