Semi-Lagrangian splitting schemes in multiple dimensions & exponential integrators

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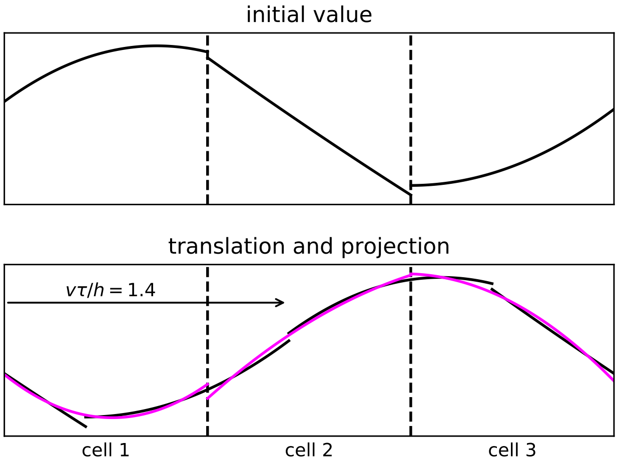
Link to slides: `http://www.einkemmer.net/training.html`

Semi-Lagrangian discontinuous Galerkin approximation

Semi-Lagrangian discontinuous Galerkin approximation

Idea: Translation and projection to a subspace of piecewise polynomials (without any continuity constraint).



J.A. Rossmanith, D.C. Seal, J. Comput. Phys. (2011)
N. Crouseilles, M. Mehrenberger, F. Vecil, ESAIM: Proceedings (2011)
L.E., A. Ostermann, SIAM J. Numer. Anal. (2014)

Technical details

We use the approximation

$$\tilde{u}(x) = \sum_{i=0}^{N-1} \sum_{j=0}^{d} \tilde{u}_{ij} \ell_{ij}(x), \qquad \ell_{ij} \text{ Lagrange polynomials in } i\text{th cell.}$$

Since

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{u}^{n+1}(x)\varphi(x)\,\mathrm{d}x = \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{u}^{n}(x-a\Delta t)\varphi(x)\,\mathrm{d}x = \int_{x_{i-1/2}-a\Delta t}^{x_{i+1/2}-a\Delta t} \tilde{u}(x)\varphi(x+a\Delta t)\,\mathrm{d}x$$

we can choose $\alpha \in [0,1)$ such that $x_{i-1/2} - a\Delta t = x_{i^*-1/2} + \alpha\Delta x$ for some $i^\star$.

Then

$$\begin{aligned}\sum_{j'} \tilde{u}_{ij'}^{n+1} \int_{x_{i-1/2}}^{x_{i+1/2}} \ell_{ij'}(x)\ell_{ij}(x)\,\mathrm{d}x &= \sum_{j'} \tilde{u}_{i^*j'}^{n} \int_{x_{i^*-1/2}+\alpha\Delta x}^{x_{i^*+1/2}} \ell_{i^*j'}(x)\ell_{ij}(x+a\Delta t)\,\mathrm{d}x \\ &+ \sum_{j'} \tilde{u}_{i^*+1;j'}^{n} \int_{x_{i^*+1/2}}^{x_{i^*+1/2}+\alpha\Delta x} \ell_{i^*+1;j'}(x)\ell_{ij}(x+a\Delta t)\,\mathrm{d}x.\end{aligned}$$

Technical details

The evaluation of the integral on the left hand side yields

$$\frac{h\omega_j}{2}\tilde{u}_{ij}^{n+1} = \sum_{j'} \tilde{u}_{i^*j'}^{n} \int_{x_{i^*-1/2}+\alpha h}^{x_{i^*+1/2}} \ell_{i^*j'}(x)\ell_{ij}(x+a\Delta t)\,\mathrm{d}x$$
$$+ \sum_{j'} \tilde{u}_{i^*+1;j'}^{n} \int_{x_{i^*+1/2}}^{x_{i^*+1/2}+\alpha h} \ell_{i^*+1;j'}(x)\ell_{ij}(x+a\Delta t)\,\mathrm{d}x,$$

Integrals can be evaluated exactly by Gauss–Legendre quadrature

$$\tilde{u}_{ij}^{n+1} = \sum_{j'} A_{jj'}^{\alpha}\tilde{u}_{i^*j'}^{n} + \sum_{j'} B_{jj'}^{\alpha}\tilde{u}_{i^*+1;j'}^{n}.$$

Semi-Lagrangian discontinuous Galerkin approximation

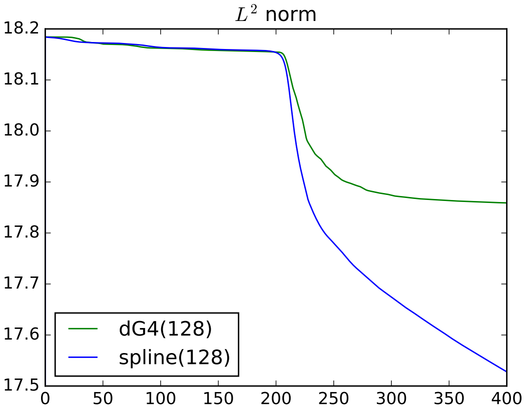
The semi-Lagrangian discontinuous Galerkin scheme

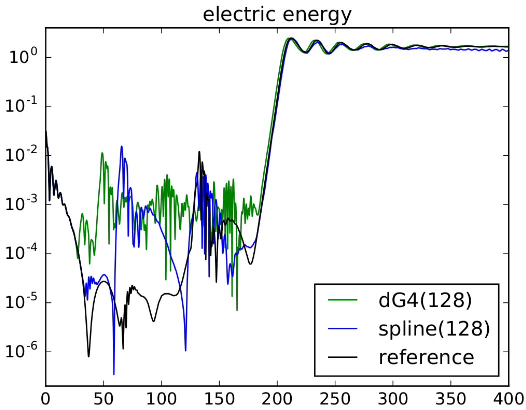
- is **mass and momentum conservative**
- introduces **little numerical diffusion**
- is a **local method** (only requires data from two adjacent cells)
- completely explicit (**no linear solves**)

Well suited for **parallelization**.

Two-stream instability

L^2 **norm** as a measure of **numerical diffusion**.

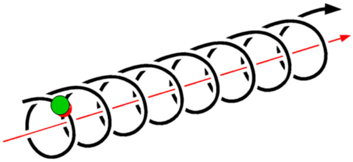




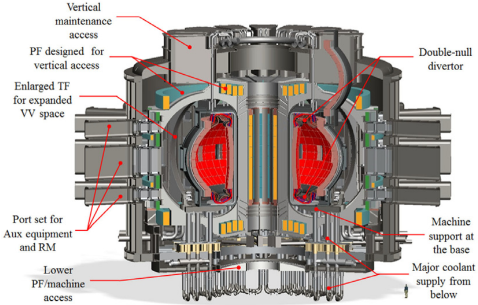
Drift-kinetic equation

Strongly magnetized plasmas

In fusion applications we have **strongly magnetized** plasmas



$$F = q(E + \mathbf{v} \times \mathbf{B})$$



Gyrokinetics averages over the motion perpendicular to the magnetic fields.

- Reduces the problem to five dimensions (3 in space and 2 in velocity).
- Removes the extremely fast gyromotion from the model (order of ps for electrons).

Picture from doi:10.1088/0029-5515/55/5/053027 and Matthias Hirsch (CC).

Particle in a magnetic field

Physical situation: particle in a strong constant magnetic field B.

Newton's equation of motion

$$m\ddot{x} = q\dot{x} \times B, \qquad B = Be_z$$

with solution

$$x = x_0 + \rho \sin \omega_c t, \qquad y = y_0 + \rho \cos \omega_c t, \qquad z = z_0 + tv_{\|}.$$

$$\text{Larmor radius} \quad \rho = \frac{mv_\perp}{|q|B}, \qquad \text{Cyclotron frequency} \quad \omega_c = \frac{|q|B}{m}.$$

Complete solution can be written as

$$\dot{x} = v = v_{\|} e_z + v_g$$

Typically in fusion applications $\rho \approx$ mm and $\omega_c \approx$ Ghz.

Particle in a magnetic field

Consider

$$m\ddot{x} = q\dot{x} \times B + F$$

with solution

$$\dot{x} = v = v_{\parallel} e_z + v_g + \frac{F \times B}{qB^2}, \qquad \ddot{x} = a = \frac{F_z}{m} e_z.$$

If we **average** over v_g we get

$$\overline{v} = v_{\parallel} e_z + \frac{F \times B}{qB^2},$$

where the second term is called a **drift**.

Let us be more specific:

- Magnetic field points along the z direction, i.e. $B = |B| e_z$
- Force F given by an electric field, i.e. $F = qE$.

Drift-kinetic equation

As usual we perform a nondimensionalization. Here effectively $|B| = 1$, $q = 1$, $m = 1$.

For the velocity and acceleration we have

$$v = v_{\|} e_z + E \times e_z = \begin{bmatrix} E_y \\ -E_x \\ v_{\|} \end{bmatrix}, \qquad a = \begin{bmatrix} 0 \\ 0 \\ E_z \end{bmatrix}.$$

From conservation of phase space

$$\partial_t f + v \cdot \nabla_x f + a \cdot \nabla_v f = 0$$

we obtain the **drift-kinetic equation**

$$\partial_t f + E_y \partial_x f - E_x \partial_y f + v \partial_z f + E_z \partial_v f = 0$$

with $f(t, x, y, z, v = v_{\|})$ coupled to some equation for E.

Drift-kinetic equation

We now have a 3+1 dimensional system

$$\partial_t f + E_y \partial_x f - E_x \partial_y f + v \partial_z f + E_z \partial_v f = 0$$

that we need to solve for $f(t, x, y, z, v)$.

Use the same splitting as before?

The **major difference** is that for $E_y \partial_x f$ and $E_x \partial_y f$ the speed of advection (E_y and $-E_x$, respectively) is **not independent of the direction of advection.**

Conservation of mass

Mass

$$\mathcal{M} = \int f(t,x,v)\,d(x,v).$$

is a conserved quantity.

Proof: We have

$$\partial_t \mathcal{M} = \int \partial_t f\,d(x,v) = \int (E_y \partial_x f - E_x \partial_y f)\,d(x,v) = \int E_\perp \cdot \nabla_{xy} f\,d(x,v),$$

where $E_\perp = (E_y, -E_x)^{\mathrm{T}}$. Then

$$E_\perp \cdot \nabla_{xy} f = \nabla_{xy} \cdot (E_\perp f) - (\nabla_{xy} \cdot E_\perp) f = \nabla_{xy} \cdot (E_\perp f)$$

since $\nabla_{xy} \cdot E_\perp = E_{xy} - E_{xy} = 0$.

Finally, we have $\partial_t \mathcal{M} = \int \nabla_{xy} \cdot (E_\perp f)\,d(x,v) = 0$.

Splitting

Splitting

Drift-kinetic equation

$$\partial_t f + E_y \partial_x f - E_x \partial_y f + v \partial_z f + E_z \partial_v f = 0,$$

where $A_1 = -E_y \partial_x f$, $A_2 = E_x \partial_y f$, $A_3 = -v \partial_z f$, and $A_4 = -E_z \partial_v f$.

Lie splitting:

$$f^{n+1} = \mathrm{e}^{\Delta t A_4} \mathrm{e}^{\Delta t A_3} \mathrm{e}^{\Delta t A_2} \mathrm{e}^{\Delta t A_1} f^n.$$

But computing e.g. $\mathrm{e}^{\Delta t A_1} f^n$ we need to solve

$$\partial_t f + E_y \partial_x f = 0, \qquad f(0, x, v) = f^n(x, v).$$

This is, in general, **not mass conservative** (conservation only holds if $\partial_x E_y = 0$.)

A different splitting

Drift-kinetic equation

$$\partial_t f + E_y \partial_x f - E_x \partial_y f + v \partial_z f + E_z \partial_v f = 0$$

where $A_1 = -E_y \partial_x f + E_x \partial_y f$, $A_2 = -v \partial_z f$, and $A_3 = -E_z \partial_v f$.

Lie splitting:

$$f^{n+1} = \mathrm{e}^{\Delta t A_3} \mathrm{e}^{\Delta t A_2} \mathrm{e}^{\Delta t A_1} f^n.$$

Good:

- one-dimensional advection, usually no stringent CFL condition.
- one-dimensional advection with stringent CFL condition.

Challenge:

- Two-dimensional advection with coefficients that depend on the direction of advection.

Strang splitting

Predictor:

- Compute a **Lie splitting step** with step size $\Delta/2$ to obtain $f^\star$.
- Compute $E^\star$ from $f^\star$.

Corrector:

- Solve $1D$ advection $\partial_t f + v\partial_z f = 0$ for $\Delta t/2$ with semi-Lagrangian method.
- Solve $1D$ advection $\partial_t f + E_z^\star f \partial_v f = 0$ for $\Delta t/2$ with semi-Lagrangian method.
- Solve **2D advection** $\partial_t f + E_y^\star \partial_x f - E_x^\star \partial_y f = 0$ for Δt.
- Solve $1D$ advection $\partial_t f + E_z^\star f \partial_v f = 0$ for $\Delta t/2$ with semi-Lagrangian method.
- Solve $1D$ advection $\partial_t f + v\partial_z f = 0$ for $\Delta t/2$ with semi-Lagrangian method.

This results in a **second order** method.

Total of 5 1D advections and 2 2D advection

- Reuse first step in predictor for the corrector.

V. Grandgirard et al. J. Comput. Phys. 217:395–423, 2006.

Two-dimensional semi-Lagrangian schemes

Semi-Lagrangian schemes

We now have to solve for the characteristic curves

$$\dot{X}(t) = -E_y(t, X(t), Y(t)), \qquad \dot{Y}(t) = E_x(t, X(t), Y(t)).$$

with

$$X(\Delta t) = x, \qquad Y(\Delta t) = y.$$

Nonlinear system of differential equations.

Even neglecting the space discretization error we only get e.g. conservation of mass if the characteristics are determined exactly.

Spline interpolation

Option 1: Compute 1D spline interpolation at each grid point in y

$$u(x, y_1), \quad \dots, \quad u(x, y_{n_y}).$$

The coefficients can be precomputed.

Then perform a 1D spline interpolation to obtain an approximation of $u(x, y)$.

- **This has to be repeated for each characteristic curve (expensive).**

Spline interpolation

Option 2: The function can be approximated by a 2d cubic B-spline interpolation

$$u(x,y) = \sum_{ij} c_{ij} \Lambda_i(x) \Lambda_j(y),$$

where Λ_i is the 1D spline basis.

The coefficients can then be determined by solving a sparse $n_x n_y \times n_x n_y$ linear system.

- **Tridiagonal structure of the 1D case is lost.**

V. Grandgirard et al. J. Comput. Phys. 217:395–423, 2006.
N. Crouseilles et al. Eur. Phys. J. D. 68:252, 2014.

Spline interpolation

In both cases **mass conservation is lost.**

This is a general problem with backward semi-Lagrangian schemes for non-constant advection.

- **Even if the characteristics are solved exactly.**

Semi-Lagrangian discontinuous Galerkin approximation

We want to solve

$$\partial_t f + \partial_x(a_1(x,y)f) + \partial_y(a_2(x,y)f) = 0, \qquad f(0,x,y) = g(x,y)$$

using a **two-dimensional semi-Lagrangian discontinuous Galerkin** approximation.

The adjoint problem is given by

$$\partial_t \Phi + a_1(t,x,y)\partial_x \Phi + a_2(t,x,y)\partial_y \Phi = 0$$

with

$$\Phi(t^{n+1},x,y) = \varphi(x,y),$$

where φ is a test function.

X. Cai, W. Guo, J.-M. Qiu. J. Sci. Comput. 73:514–542, 2017.

Semi-Lagrangian discontinuous Galerkin approximation

Since

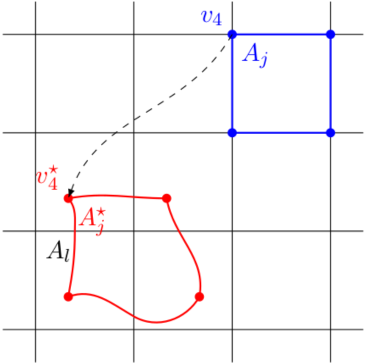
$$\partial_t(f\Phi) = (\partial_x(a_1 f) + \partial_y(a_2 f))\,\Phi + a_1 f \partial_x \Phi + a_2 f \partial_y \Phi$$

we have

$$\partial_t \int_{A_j(t)} f(t,x,y)\Phi(t,x,y)\,d(x,y) = 0.$$

Evaluating this equation at time t^n and t^{n+1} gives

$$\int_{A_j} f^{n+1}\varphi(x,y)\,d(x,y) = \int_{A_j^\star} f^n \Phi(t^n,x,y)\,d(x,y).$$



Picture courtesy of Jing-Mei Qiu (doi:10.1007/s10915-017-0554-0).

Semi-Lagrangian discontinuous Galerkin approximation

Plugging in the basis functions $\varphi_{j,ab}$ we get

$$\int_{A_j} f^{n+1}\varphi_{j,ab}(x,y)\,d(x,y) \approx \int_{A_j^\star} f^n \varphi^\star_{j,ab}(t^n,x,y)\,d(x,y)$$

Then

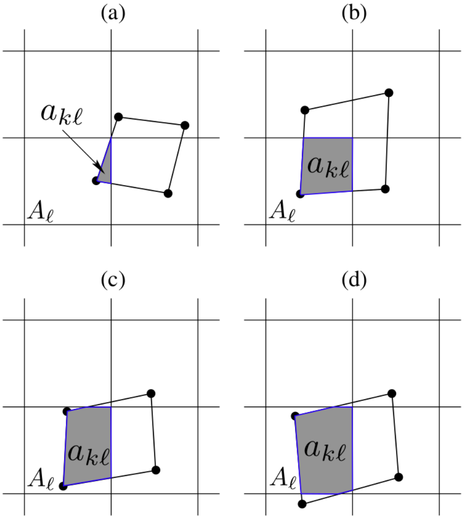
$$\begin{aligned}
f^{n+1}_{j,ab}\omega_{ab} &\approx \int_{A_j^\star} f^n \varphi^\star_{j,ab}(x,y)\,d(x,y) \\
&\approx \sum_l \int_{a_{jl}} f^n \varphi^\star_{j,ab}(x,y)\,d(x,y) \\
&\approx \sum_l \sum_{r,s} f^n_{l,rs} \int_{a_{jl}} \varphi_{l,rs}(x,y)\varphi^\star_{j,ab}(x,y)\,d(x,y) \\
&\approx \sum_l \sum_{r,s} f^n_{l,rs} \sum_{u,v} C^{uv}_{lrsjab} \int_{a_{jl}} x^u y^v\,d(x,y)
\end{aligned}$$

Semi-Lagrangian discontinuous Galerkin approximation

We now have to integrate monomials over polygons.

An error due to approximating the boundary is committed.

Mass is exactly preserved as boundaries from different cells match.



Picture courtesy of Jing-Mei Qiu (doi:10.1007/s10915-017-0554-0).

Semi-Lagrangian discontinuous Galerkin approximation

Green's theorem states that.

$$\int_A \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) d(x,y) = \int_{\partial A} P dy - Q dx.$$

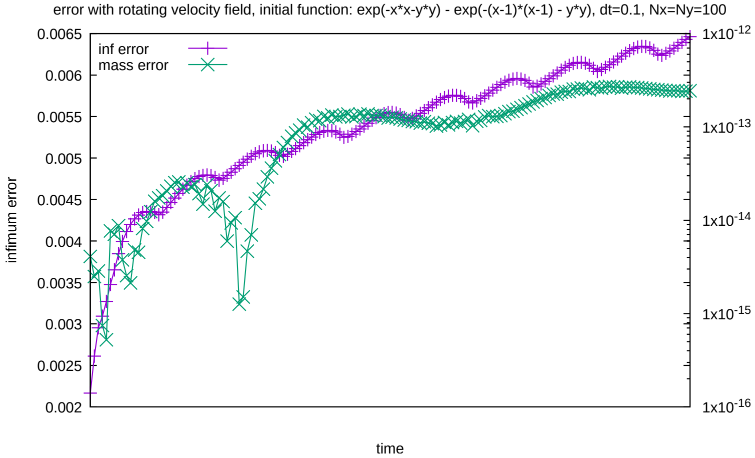
By choosing $Q = 0, P = \frac{x^{u+1} y^v}{u+1}$, we get

$$f^{n+1}_{j,ab} \omega_{ab} \approx \sum_l \sum_{r,s} f^n_{l,rs} \sum_{u,v} C^{uv}_{lrsjab} \sum_{z \in \partial A^\star_{j,l}} \int_z \frac{x^{u+1} y^v}{u+1} dy$$

The **integral can be computed exactly by a Gaussian quadrature rule**.

Numerical results

Rigid body rotation: $\partial_t f - \partial_x(yf) + \partial_y(xf) = 0.$



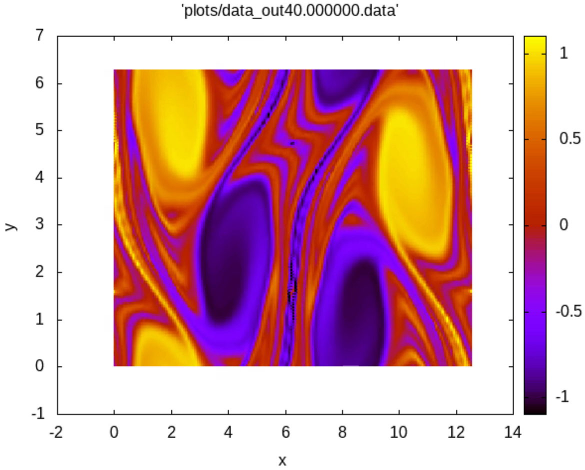
Numerical results

Guiding center equations:

$$\partial_t f + \nabla \cdot (\mathbf{E}^\perp f) = 0$$

with

$$\mathbf{E}^\perp = (E_2, -E_1).$$
$$\mathbf{E} = -\nabla \phi$$
$$-\Delta \phi = f.$$



Splitting with an explicit part

Splitting

Drift-kinetic equation

$$\partial_t f + E_y \partial_x f - E_x \partial_y f + v \partial_z f + E_z \partial_v f = 0.$$

In many cases the condition $v\Delta t < \Delta x$ dominates $\|E\|_\infty \Delta t < \Delta x$.

- E.g. in situations where we are close to the linear regime.

Idea: instead of computing $\mathrm{e}^{\Delta t A_1} f^n$ using a semi-Lagrangian scheme we perform an explicit approximation. E.g. for explicit Euler we get

$$f^{n+1} = \mathrm{e}^{\Delta t A_3} \mathrm{e}^{\Delta t A_2} (I + \Delta t A_1) f^n.$$

Easy to find mass conservative schemes: Lax–Wendroff, RK4+CD, ...

- But higher order methods require many subflows (especially if we split into more than two parts).

Exponential integrators

Exponential integrators

Drift-kinetic equation

$$\partial_t f + E_y \partial_x f - E_x \partial_y f + v \partial_z f + E_z \partial_v f = 0.$$

In many problems $v\partial_z f$ dictates the CFL condition.

Idea: treat the blue part explicitly and the red part using a semi-Lagrangian scheme.

N. Crouseilles, L.E., M. Prugger, Comput. Phys. Commun. 224 (2018).

Exponential integrators

Our problem can be written as

$$\partial_t f = Af + F(f),$$

where

$$Af = -v\partial_z f, \qquad F(f) = -E_y\partial_x f + E_x\partial_y f - E_z\partial_v f.$$

Exponential integrators are based on the variation of constants formula

$$f(t^n + \Delta t) = \exp(\Delta t A) f(t^n) + \int_0^{\Delta t} \exp((\Delta t - s)A) F(f(t^n + s))\, \mathrm{d}s.$$

We approximate the integral to obtain the **exponential Euler method**

$$f(t^n + \Delta t) \approx f^{n+1} = \exp(\Delta t A) f^n + \Delta t \varphi_1(\Delta t A) F(f^n),$$

where $\varphi_1(z) = (\mathrm{e}^z - 1)/z$ is an entire function (similar to the exponential).

Lawson methods

We perform a change of variables

$$g(t) = \exp(-tA)f(t)$$

to obtain

$$\partial_t g = \exp(-tA)F(\exp(tA)g(t)).$$

Now we apply the **explicit Euler scheme** to the transformed equation

$$g(t^n + \Delta t) \approx g^{n+1} = g^n + \Delta t \exp(-t^n A)F(\exp(t^n A)g^n).$$

Reversing the change of variables yields the **Lawson–Euler method**

$$f^{n+1} = \exp(\Delta t A)\left(f^n + \Delta t F(f^n)\right).$$

We note that

- the underlying Runge–Kutta method uniquely determines the Lawson scheme;
- only exponentials are required (convenient for semi-Lagrangian solvers);
- Lawson methods can be considered a subset of exponential integrators.

Second order Lawson scheme

We start with the classic midpoint rule applied to $\partial_t g = G(t, g)$

$$k_1 = g^0 + \frac{\Delta t}{2} G(0, g^0)$$
$$g^1 = g^0 + \Delta t G(\Delta t/2, k_1)$$

Plugging in $G(t, g) = \exp(-tA) F(\exp(tA) g)$ we get

$$k_1 = g^0 + \frac{\Delta t}{2} F(g^0)$$
$$g^1 = g^0 + \Delta t \exp(-\Delta t/2A) F(\exp(\Delta t/2A) k_1)$$

Since $g^0 = f^0$ and $g^1 = \exp(-\Delta t A) f^0$ we then have

$$\bar{k}_1 = \exp(\Delta t/2A) \left(f^0 + \frac{\Delta t}{2} F(f^0) \right)$$
$$f^1 = \exp(\Delta t A) f^0 + \Delta t \exp(\Delta t/2A) F(\bar{k}_1)$$

Second order exponential methods

Second order two stage exponential integrator

$$
\begin{aligned}
k_1 &= e^{\Delta t A} f^n + \Delta t \varphi_1(\Delta t A) F(f^n) \\
f^{n+1} &= e^{\Delta t A} f^n + \Delta t \Big[(\varphi_1(\Delta t A) - \varphi_2(\Delta t A)) F(f^n) + \varphi_2(\Delta t A) F(k_1) \Big]
\end{aligned}
$$

with $\varphi_1(z) = (e^z - 1)/z$ and $\varphi_2(z) = (e^z - 1 - z)/z^2$.

Can be made more efficient by writing

$$
\begin{aligned}
k_1 &= f^n + \Delta t \varphi_1(\Delta t A) \left(F(f^n) + A f^n \right) \\
f^{n+1} &= f^n + \Delta t \Big[(\varphi_1(\Delta t A) \left(F(f^n) + A f^n \right) + \varphi_2(\Delta t A) \left(F(k_1) - F(f^n) \right) \Big]
\end{aligned}
$$

This requires **two matrix functions** and **two evaluations of F** per step.

M. Hochbruck, A. Ostermann, Acta Numer. (2010).

Exponential methods

We still have to satisfy the CFL condition imposed by $\partial_t f = F(f)$.

- **No time step restriction from the linear part** $\partial_t f = Af$.

More efficient to reach higher order in cases where we split into more than two parts.

Lawson methods are more convenient in a semi-Lagrangian setting (only exponentials).

Well known that Lawson methods suffer from order reduction in case of non-trivial boundary conditions.

M. Hochbruck, J. Leibold, A. Ostermann. Numer. Math. 145:553–580, 2020.

Conservation

Theorem

Lawson schemes ***conserve all linear invariants*** *that are conserved by* $\partial_t f = Af$ *and* $\partial_t f = F(f)$.

Proof.

Let us write the linear invariant as If. Then from the assumption we have

$$0 = \partial_t(If) = I\partial_t f = IAf, \qquad \text{and by a similar argument} \qquad 0 = IF(f).$$

In transformed variables we have

$$\partial_t(Ig) = I(\partial_t g) = I\exp(-tA)F(\exp(tA)g(t)) = IF(\exp(tA)g(t)) = 0.$$

This implies conservation since a Runge–Kutta scheme preserves linear invariants and

$$If = I\exp(tA)g = Ig.$$

□

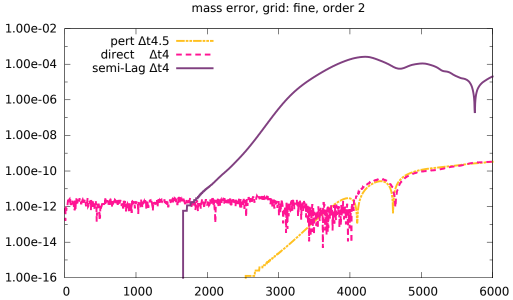
Conservation

A similar result can be obtained for exponential Runge–Kutta methods.

In particular, this implies **mass and momentum conservation** (assuming an appropriate space discretization is used).

Note that we have not considered space discretizaton.

- Difficult term is treated explicitly.
- E.g. Arakawa's method provides both mass and momentum conservation.



L.E. A. Ostermann. J. Comput. Phys. 299:716–730, 2015.
N. Crouseilles, L.E., M. Prugger. Compu. Phys. Commun. 224:144-153, 2018.
A. Arakawa. J. Comput. Phys. 1:1, 1966.

The orthodoxy

Stability of exponential integrators is well understood

- CFL condition dictated by the nonlinear part.
- Rigorous convergence results for parabolic problems.

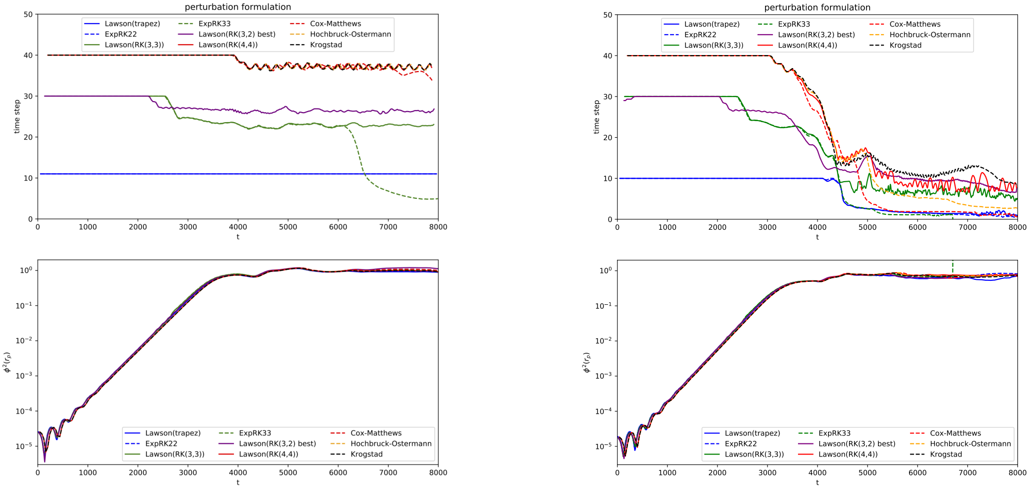
Lawson methods should be avoided

- Order reduction for problems with non-trivial boundary conditions.
- If iterative methods are used (e.g. Krylov or Leja), the φ_k functions converge faster than the exponential.

M. Hochbruck, A. Ostermann, Acta Numer. (2010).
M. Hochbruck, J. Leibold, A. Ostermann, Numer. Math. 145 (2020).

Ion temperature gradient (ITG) instability

Coarse resolution left and fine resolution right for an ion ITG instability.



We use an automatic step-size controller.

Stability of exponential methods

There are significant **surprises with respect to the stability of exponential methods for hyperbolic problems**.

Numerical observation indicate that **Lawson methods are significantly more stable than exponential integrators**.

Exponentials are easier to deal with when using semi-Lagrangian schemes.

Linear analysis

The following **test equation** provides insight into the issue of **stability**

$$\partial_t f = iaf + \lambda f, \qquad a \in \mathbb{R},\ \lambda \in \mathbb{C}.$$

For **Lawson methods** we perform the change of variables

$$g(t) = e^{-iat} f(t).$$

to obtain

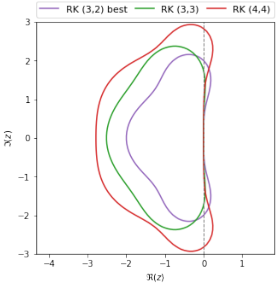
$$\partial_t g = e^{-iat}\lambda(e^{iat} g) = \lambda g.$$

Apply an explicit RK method with stability function ϕ

$$g^{n+1} = \phi(z) g^n, \qquad z = \lambda \Delta t,$$

Reversing the change of variables

$$f^{n+1} = e^{ia\Delta t}\phi(z) f^n.$$



Condition for stability: $|\Phi(z)| = |e^{ia\Delta t}\phi(z)| = |\phi(z)| \leq 1.$

Exponential integrators

For the **ExpRK22** scheme applied to $\partial_t f = iaf + \lambda f$ with $f^n = 1$ we have

$$k_1 = e^{ia\Delta t} + \Delta t \varphi_1(ia\Delta t)\lambda$$
$$f^{n+1} = e^{ia\Delta t} + \Delta t\Big[(\varphi_1(ia\Delta t) - \varphi_2(ia\Delta t))\lambda + \varphi_2(ia\Delta t)\lambda k_1\Big]$$

with $\varphi_1(z) = (e^z - 1)/z$ and $\varphi_2(z) = (e^z - 1 - z)/z^2$.

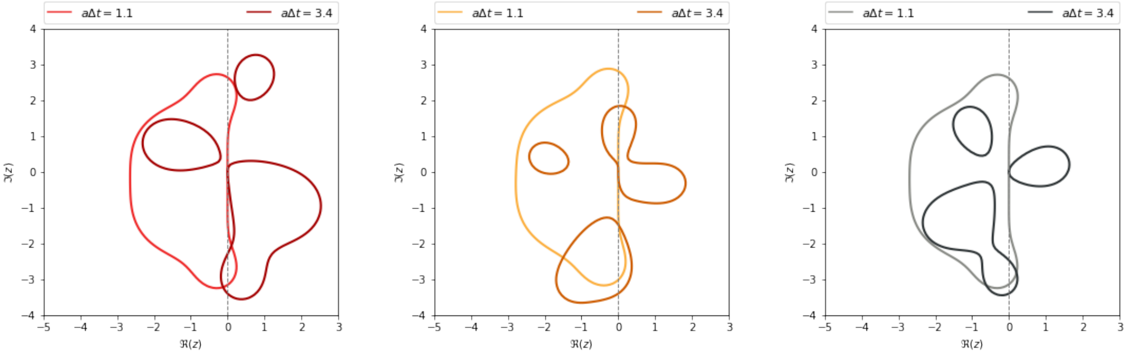
The **stability function** Φ is then given by

$$\Phi(z) = e^{ia\Delta t} + \Big(\varphi_1(ia\Delta t) - \varphi_2(ia\Delta t) + e^{ia\Delta t}\varphi_2(ia\Delta t)\Big)z + \varphi_1(ia\Delta t)\varphi_2(ia\Delta t)z^2.$$

The **limit** $a \to 0$ yields $\Phi(z) = 1 + z + z^2/2$.

Exponential integrators

Stability domain for different advection speeds a and **Cox–Matthews** (left), the **Hochbruck & Ostermann** scheme (middle), and the **Krogstad** scheme (right).



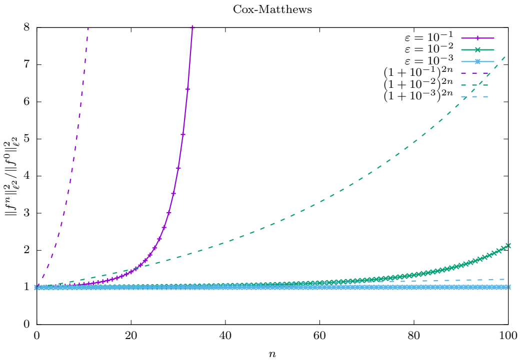
Even for moderate a, stability on the imaginary axis is terrible.

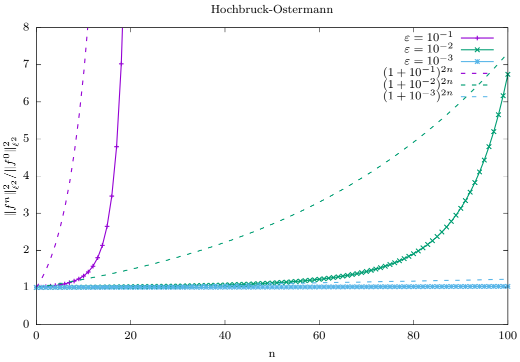
Why do exponential integrators work at all?

The **classic concept of stability** is **overly restrictive**.
Behavior of numerical results can be explained by introducing the ϵ-stability domain

$$\mathcal{D}_\varepsilon = \{z \in \mathbb{C} : |\phi(z)| \leq 1 + \varepsilon\}.$$

- For finite time step sizes a small $\epsilon > 0$ can still result in acceptable results.





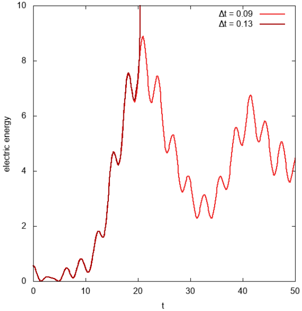
How good is the estimate

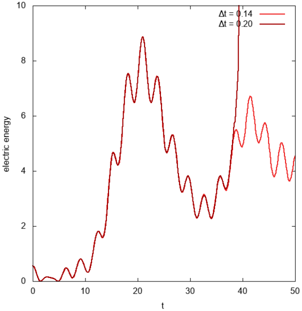
Utility of $\partial_t f = iaf + \lambda f$ is unclear as it only applies to problems of the form

$$\partial_t f = Af + Bf, \qquad [A, B] = 0.$$

The **theory** provides a **necessary condition**.

Good agreement even for nonlinear problems.
Vlasov equation that simulates a bump on tail instability.
Lawson(RK(4,4))+WENO5 (left) and centered differences (right).





N. Crouseilles, L.E., J. Massot, J. Comput. Phys. 420 (2020).

Literature

Literature

[V. Grandgirard et al. J. Comput. Phys. 217:2, 2006]
[N. Crouseilles et al. Eur. Phys. J. D 68, 2014]

- Splitting/semi-Lagrangian schemes for the drift-kinetic equations (including spline based).

[X. Cai, W. Guo, J.-M. Qiu. J. Sci. Comput. 73, 2017]

- Two-dimensional semi-Lagrangian discontinuous Galerkin scheme.

[J.D. Lawson. SIAM J. Numer. Anal. 4:3, 1967]

- The original paper on Lawson schemes.

[M. Hochbruck, A. Ostermann. Acta Numer. 19, 2010]

- Review article on exponential integrators.

Literature

[N. Crouseilles, L.E., M. Prugger, Comput. Phys. Commun. 224 (2018)]

- Exponential integrators/Arakawa for the drift-kinetic equations.

[N. Crouseilles, L.E., J. Massot, J. Comput. Phys. 420 (2020)]

- Lawson schemes and stability analysis for the drift-kinetic equations.

[M. Hochbruck, J. Leibold, A. Ostermann. Numer. Math. 145, 2020]

- Analysis of order reduction for Lawson schemes in case of non-trivial boundary conditions.