Time integration, convergence, and the importance of conservation

Lukas Einkemmer
University of Innsbruck

Dobbiaco summer school, 2022

Link to slides: `http://www.einkemmer.net/training.html`

Generic methods for conservation laws

Conservative formulation

The **Vlasov equation (in advection form)**

$$\partial_t f + v \cdot \nabla_x f + F(f) \cdot \nabla_v f = 0$$

can be easily written as a conservation law

$$\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot (F(f) f) = 0$$

assuming that the force F is only x-dependent (e.g. Vlasov–Poisson).

Then we have a **scalar conservation law**

$$\partial_t f + \nabla_{x,v} \cdot G(f) = 0, \qquad G(f) = \begin{bmatrix} v \\ F(f) \end{bmatrix} f.$$

Upwind

The classic **conservative upwind discretization** (for simplicity in 1+1d) is

$$\frac{f_{ij}^{n+1} - f_{ij}^{n}}{\Delta t} = -v^{+} D_x^{-}(f^n)_{ij} - v^{-} D_x^{+}(f^n)_{ij} - (F^n)^{+} D_v^{-}(f^n)_{ij} - (F^n)^{-} D_v^{+}(f^n)_{ij}$$

with

$$D_x^{+}(g)_{ij} = \frac{g_{i+1,j} - g_{ij}}{\Delta x}, \qquad D_x^{-}(g)_{ij} = \frac{g_{ij} - g_{i-1,j}}{\Delta x}$$

and

$$D_v^{+}(g)_{ij} = \frac{g_{i,j+1} - g_{ij}}{\Delta v}, \qquad D_v^{-}(g)_{ij} = \frac{g_{ij} - g_{i,j-1}}{\Delta v},$$

where $h^{+} = \max\{h, 0\}$ and $h^{-} = \min\{h, 0\}$.

It is **easy to determine the direction of flow**.

The scheme is **shock capturing**, i.e. it produces correct approximations to discontinuous solutions.

Why not explicit schemes

This is an **explicit scheme** with the **hyperbolic CFL conditions**

$$v\Delta t \leq \Delta x \qquad \text{and} \qquad \|F\|_\infty \Delta t < \Delta v$$

required for stability.

In particular, if we truncate velocity space as $v \in [-v_{\max}, v_{\max}]$ we have

$$v_{\max}\Delta t \leq \Delta x.$$

In many situations we have $v_{\max} \gg 1$ and we thus have a **separation of scales between the fastest velocity in the system and the bulk velocity** (thermal velocity).

For many of the classic test problems we usually choose $v_{\max} = 6 - 10$ but the ratio can be much larger

- Electron vs ion time scales
- timescales induced by external magnetic fields

Why not explicit schemes

For moderate to fine resolutions **explicit schemes suffer from the CFL condition**.

Nonlinear Landau damping: total time of computation for different methods with respect to the number of points points

Numerical method	32×32 points	32×64 points	32×128 points
FBM	03.33 s	05.39 s	10.80 s
PFC	03.56 s	06.28 s	11.20 s
FDM	17.22 s	35.27 s	71.20 s
SPECTRAL	04.10 s	08.25 s	16.90 s
CIP	13.83 s	21.40 s	43.24 s
SL SPLINE	06.12 s	10.55 s	20.90 s
SL HERMITE	03.60 s	06.90 s	11.00 s

F. Filbet, E. Sonnendrücker. Comput. Phys. Commun. 150:247–266, 2003.

Why not shock capturing schemes

The **price** to pay **for shock capturing is numerical diffusion**.

Can be analyzed by the **method of modified equations**. The upwind scheme ($v > 0$)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -v \frac{u_i^n - u_{i-1}^n}{\Delta x}$$

discretizes

$$\partial_t u + v \partial_x u = 0$$

up to first order in both time and space.

But it is a **second order approximation** to the **modified equation**

$$\partial_t u + v \partial_x u = \frac{v \Delta x (1 - \alpha)}{2} \partial_{xx} u, \qquad \alpha = v \frac{\Delta t}{\Delta x}.$$

We **add diffusion** with strength proportional to Δx.

Why not shock capturing schemes

High resolution schemes converge with higher order away from discontinuities.

- Godunov's theorem states that **a linear scheme that is monotone is at most first order accurate.**
- High resolution schemes are by necessity nonlinear (limiters, WENO, ...).

Such methods detect where the solution has large derivatives and reduce the order accordingly.

But there are negative results. E.g.

- **A total variation diminishing scheme degenerates to first order accuracy at extreme points of the solution.**

We have to choose between as little diffusion as possible and the ability to treat very sharp gradients (or shocks) in the solution.

Solution of the Vlasov equation

Consider

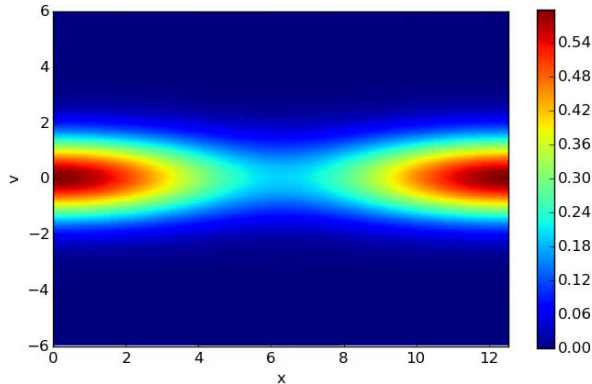
$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = 0$$
$$f(0,x,v) = e^{ikx} e^{-v^2}$$

which we can easily solve to obtain

$$f(t,x,v) = e^{ik(x-vt)} e^{-v^2}$$
$$= e^{ikx} e^{-ikvt} e^{-v^2}$$

This small scale structure in phase space is called **filamentation**.

Nonlinear Landau damping



The solution of the Vlasov equation is typically **highly-oscillatory** (many maxima and minima). Not smooth regions separated by shocks as we typically see in nonlinear conservation laws. Theory tells us that the solution is $\mathcal{C}^\infty$.

Why not implicit methods

In **stiff and highly oscillator problems implicit methods** of Gauss–Legendre type are commonly used.

- Crank–Nicolson/trapezoidal rule as the second order variant.

How to **solve the nonlinear system of equations efficiently** is a major concern.

We have to do this in a **high-dimensional setting**

- Four dimensional: $n = 256$ requires 34 GB memory
- Five dimensional: $n = 128$ requires 257 GB memory
- Six dimensional: $n = 128$ requires 35 TB memory

to store a single snapshot of the solution.

Certainly performing e.g. an iterative schemes that needs to store multiple vectors would be extremely expensive.

The splitting scheme of Cheng & Knorr

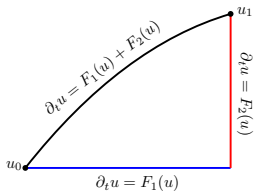
Splitting

We consider

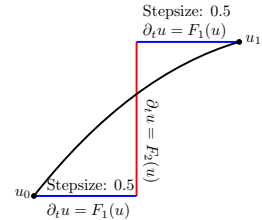
$$\partial_t u(t) = F_1(u(t)) + F_2(u(t)), \qquad u(0) = u^0,$$

where F_1 and F_2 could describe **different physics**, **timescales**, **coordinate axis, ...**

$$u^{n+1} = \varphi^{F_2}_{\Delta t} \circ \varphi^{F_1}_{\Delta t}(u^n) \textbf{ (Lie)} \qquad u^{n+1} = \varphi^{F_1}_{\Delta t/2} \circ \varphi^{F_2}_{\Delta t} \circ \varphi^{F_1}_{\Delta t/2}(u^n) \textbf{ (Strang)}$$



Lie splitting



Strang splitting

Fundamental idea of splitting is that **only subflows have to be solved**.

Time splitting

For the Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0$$

we can split as follows.

Free streaming part

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = 0$$

Solution:

$$f(\Delta t, x, v) = f(0, x - \Delta t v, v)$$

Acceleration part

$$\partial_t f(t,x,v) + F(f) \cdot \nabla_v f(t,x,v) = 0$$

Solution for constant F:

$$f(\Delta t, x, v) = f(0, x, v - \Delta t F)$$

Only translations have to be computed.

Lie splitting

But: F is not constant – nonlinear equation
Idea: Freeze force term F in the numerical algorithm.

Lie splitting: With $F^n = F(f^n)$ we obtain the algorithm

1. Compute F^n (e.g. by solving the Poisson equation).
2. Compute $f^\star(x, v) = f^n(x - \Delta t v, v)$.
3. Compute $f^{n+1}(x, v) = f^\star(x, v - \Delta t F^n)$

which is **first order** accurate.

We often write this in exponential notation as follows

$$f^{n+1} = \mathrm{e}^{\Delta t B^n} \mathrm{e}^{\Delta t A} f^n, \qquad A = -v \cdot \nabla_x, \qquad B^n = -F^n \cdot \nabla_v.$$

C.Z. Cheng, G. Knorr, J. Comput. Phys. (1976)

Strang splitting

For the Vlasov–Poisson equation we can easily extend this to **second order**

$$f^{n+1} = \mathrm{e}^{\frac{\Delta t}{2}A}\mathrm{e}^{\Delta t B^{n+1/2}}\mathrm{e}^{\frac{\Delta t}{2}A} f^n,$$

where

$$B^{n+1/2} = -E(f^\star)\cdot\nabla_v, \qquad f^\star = \mathrm{e}^{\frac{\Delta}{2}tA} f^n.$$

The main insight here is that we **freeze E at $\mathbf{t^{n+1/2}}$**.

This works since for the Vlasov–Poisson equation

$$\rho(f^{n+1/2}) \approx \int \mathrm{e}^{\frac{\Delta t}{2}B^n}\mathrm{e}^{\frac{\Delta t}{2}A} f^n \,\mathrm{d}v = \int \mathrm{e}^{\frac{\Delta t}{2}A} f^n \,\mathrm{d}v = \int f^\star \,\mathrm{d}v$$

since $\mathrm{e}^{\frac{\Delta t}{2}A}$ is just a translation in v.

Reduction to 1D

We can further split the advection term as

$$A = -v \cdot \nabla_x = -v_1 \partial_{x_1} - v_2 \partial_{x_2} - v_3 \partial_{x_3}.$$

We have

$$\mathrm{e}^{\Delta t A} = \mathrm{e}^{\Delta t A_1 + \Delta t A_2 + \Delta t A_3} = \mathrm{e}^{\Delta t A_3} \mathrm{e}^{\Delta t A_2} \mathrm{e}^{\Delta t A_1}, \qquad \text{with} \qquad A_i = -v_i \partial_{x_i}$$

since

$$[A_1, A_2] f = v_1 \partial_{x_1}(v_2 \partial_{x_2} f) - v_2 \partial_{x_2}(v_1 \partial_{x_1} f) = 0.$$

Thus, **we do not commit any further splitting error**

- Same argument holds for the acceleration term.

Reduces the Vlasov equation to a sequence of linear one-dimensional advections!

Space discretization

Now, in principle we can apply some explicit finite difference/volume/... scheme to

$$\partial_t f(t,x,v) + v_1 \cdot \nabla_{x_1} f(t,x,v) = 0.$$

in order to compute $\mathrm{e}^{\Delta t A_1} f^n$, **but that would miss the point**.

- Would be still limited by the CFL condition.

We want some **dedicated method to directly compute the translation**

$$f(0, x - \Delta t v).$$

This will be content of the next lecture.

For now let us assume that we have a black box that can compute the translation up to some space discretization error that is $\mathcal{O}(\Delta x^q)$.

Convergence

A typical convergence result is given by

$$\sup_{0 \leq n \leq N} \|f^n - f(n\Delta t)\| \leq C \left(\Delta t^p + \Delta x^q + \frac{\Delta x^q}{\Delta t} \right),$$

where n is such that $0 \leq n\Delta t \leq N\Delta t = T$ and C depends on T but is independent of Δt, Δx, and n.

Error terms

- Splitting error ($p = 1$ for Lie splitting and $p = 2$ for Strang splitting).
- Space discretization error (due to the black box)
- Term due to worst case error accumulation

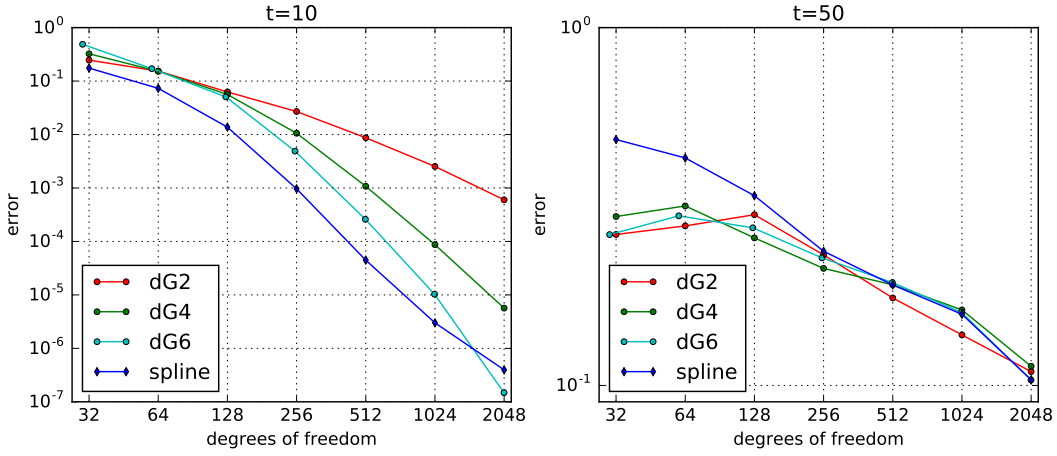
Note that $1/\Delta t \propto N$ (in practice this term is usually not an issue).

N. Besse, M. Mehrenberger. Math. Comp. 77, 2008.
L. E., A. Ostermann. SIAM J. Numer. Anal. 52:2, 2014.

The importance of conservative numerical methods

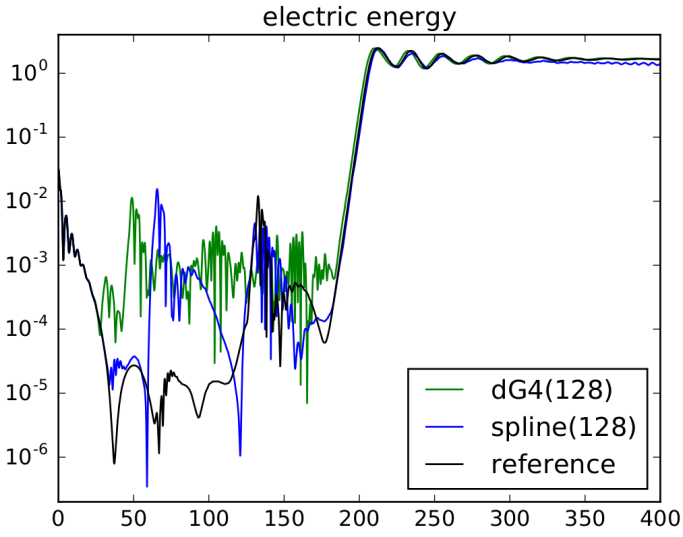
Accuracy of Vlasov simulation

Performance is often checked by using a work-precision diagram.



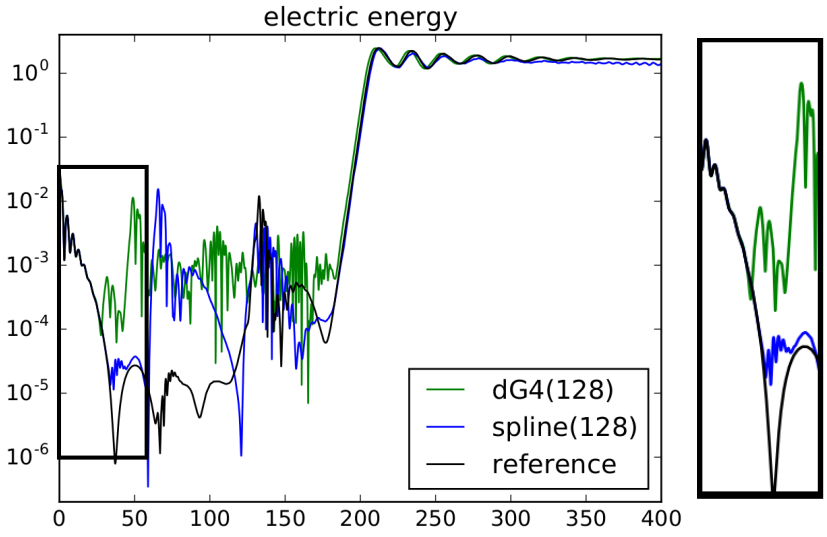
Two-stream instability

Vlasov–Poisson equation with 128^4 degrees of freedom.



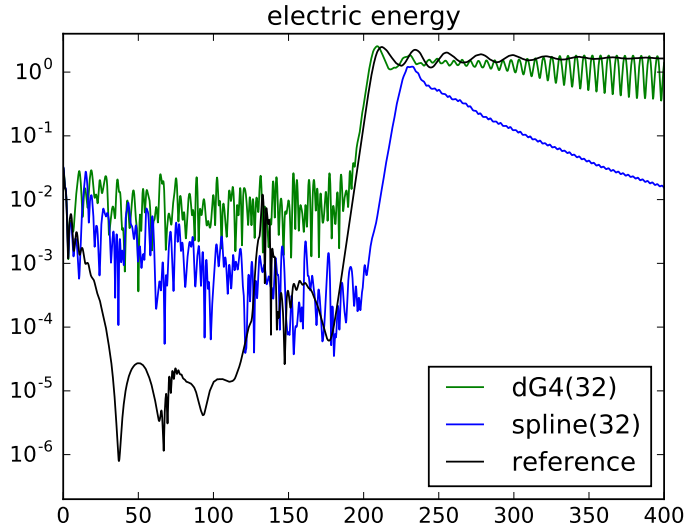
Two-stream instability

Vlasov–Poisson equation with 128^4 degrees of freedom.



Two-stream instability

Vlasov–Poisson equation with 32^4 degrees of freedom.



Discussion

We are in the **asymptotic regime** if classic convergence theory applies. That is,

$$\text{error} \leq C\left((\Delta t)^p + (\Delta x)^q\right).$$

gives a tight bound of the error.

Why is this not the case here? Consider

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) = 0, \qquad f(0,x,v) = \mathrm{e}^{ikx}\mathrm{e}^{-v^2/2}$$

which has the solution

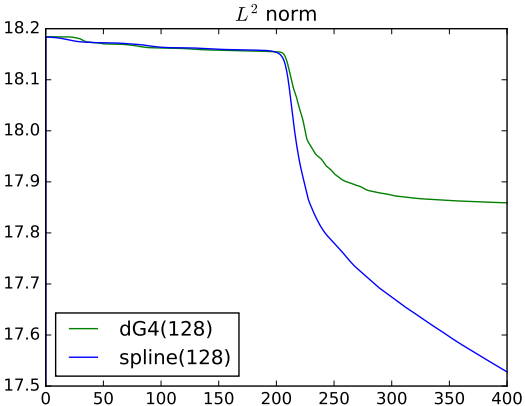
$$f(t,x,v) = \mathrm{e}^{ikx}\mathrm{e}^{iktv}\mathrm{e}^{-v^2/2}.$$

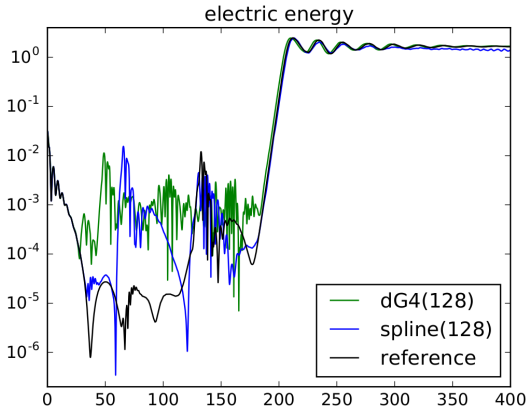
Small scale structures (e.g. filamentation, turbulence, ...) can not be fully resolved.

- **All methods are necessarily inaccurate.**
- Often we can still get good physics out of those methods.

Two-stream instability

L^2 **norm** as a measure of **numerical diffusion**.





Invariants more important than accuracy.

- Numerical methods should be designed with this in mind.

L.E. J. Comp. Phys. 376:937–951, 2019.

Moments

Physical important invariants of the Vlasov equation are mass, momentum, and energy

$$M = \int_{\Omega_v}\int_{\Omega_x} f \mathrm{d}x\mathrm{d}v, \qquad P = \int_{\Omega_v}\int_{\Omega_x} vf \mathrm{d}x\mathrm{d}v, \qquad \mathcal{E} = \frac{1}{2}\int_{\Omega_v}\int_{\Omega_x} v^2 f \mathrm{d}x\mathrm{d}v + \frac{1}{2}\int_{\Omega_x} E^2 \mathrm{d}x.$$

Associated to each global invariant is a local conservation law

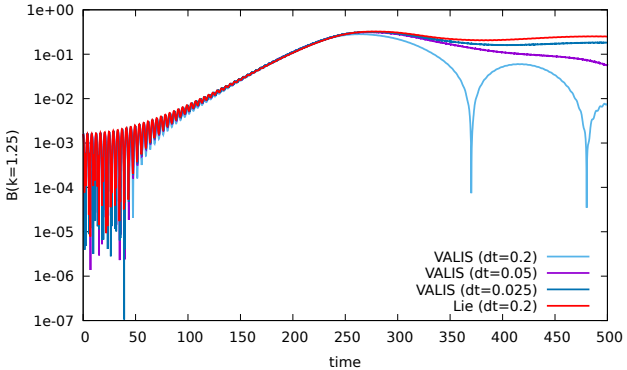
$$\partial_t \rho + \nabla \cdot j = 0, \qquad \rho = \int_{\Omega_v} f \,\mathrm{d}v, \quad j = \int_{\Omega_v} vf \,\mathrm{d}v$$

$$\partial_t j + \nabla_x \cdot \sigma = -E\rho, \qquad \sigma = \int_{\Omega_v} (v \otimes v) f \, dv$$

$$\partial_t e + \nabla_x \cdot Q = E \cdot (\partial_t E - j), \qquad e = \frac{1}{2}\int_{\Omega_v} v^2 f \, dv + \frac{1}{2}E^2, \quad Q = \frac{1}{2}\int_{\Omega_v} v v^2 f \, dv.$$

Conservation

Non-conservative numerical methods can lead to **physically wrong solutions**.

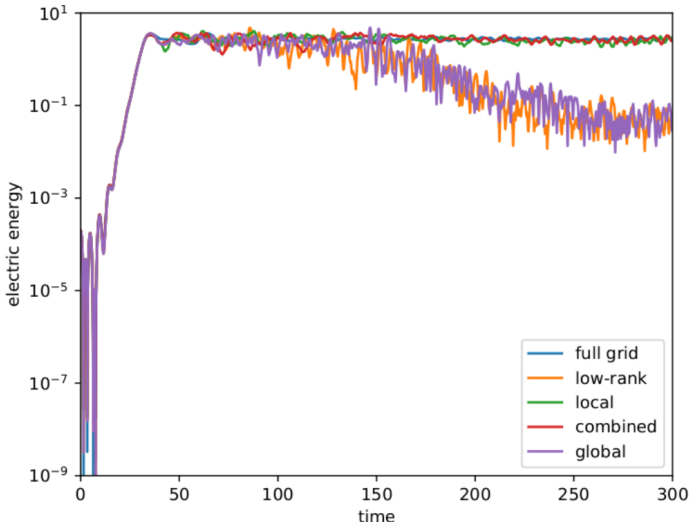


for reasonable time step sizes (here for the Weibel instability).

Conservative schemes often provide **improved qualitative properties of the solution**.

Global vs local conservation

We should not forget the **local conservation law.**



Properties of the splitting

Hamiltonian systems

Reminder: In a Hamiltonian system the time evolution of a quantity $F(p, q)$ can be written as

$$\partial_t F = \{F, H\}$$

with Hamiltonian H and the symplectic Poisson bracket

$$\{F, G\} = \nabla_q F \cdot \nabla_p G - \nabla_p F \cdot \nabla_q G.$$

Since $\partial_t H = \{H, H\} = 0$ we follow that the Hamiltonian, i.e. **the energy of the system, is conserved**.

Every quantity C such that $\{C, H\} = 0$ is conserved.

A **non-canonical Hamiltonian** system admits a Poisson bracket $[\cdot, \cdot]$ (not necessarily the symplectic bracket above) that satisfies Anticommutativity, Bilinearity, Leibniz's rule, and the Jacobi identity.

Non-canonical Hamiltonian systems

Example: The generalized Lotka–Volterra model

$$\dot{u} = u(v+w), \qquad \dot{v} = v(u-w+1), \qquad \dot{w} = w(u+v+1)$$

is a **non-canonical Hamiltonian** system with

$$H(u,v) = -u+v+w+\ln v - \ln w \quad \text{and} \quad [F,G] = (\nabla_{uvw} F)^T \underbrace{\begin{bmatrix} 0 & uv & uw \\ -uv & 0 & -vw \\ -uw & vw & 0 \end{bmatrix}}_{B} \nabla_{uvw} G$$

C is a **Casimir invariant** if $\{C, G\} = 0$ holds for all G.

- We call the Poisson bracket degenerate.

Example: For the Lotka–Volterra model $C(u,v) = -\ln u - \ln v + \ln w$ is a Casimir invariant as $(\nabla_{uvw} C)^T B = 0$.

Hamiltonian structure of Vlasov–Poisson

The **Vlasov–Poisson equations** have a **non-canonical Hamiltonian** structure.

$$\begin{aligned} H &= \frac{1}{2}\int E^2 \,\mathrm{d}x + \frac{1}{2}\int v^2 f \,\mathrm{d}(x,v) \\ &=: H_E + H_f. \end{aligned}$$

Evolution of a functional F

$$\partial_t F = [F, H] = [F, H_E] + [F, H_f]$$

with a non-canonical **Poisson bracket**

$$[F, G] = \int \frac{\delta F}{\delta f}\left\{\frac{\delta G}{\delta f}, f\right\} \mathrm{d}(x,v), \qquad \{f, g\} = \nabla_x f \cdot \nabla_v g - \nabla_v f \cdot \nabla_x g.$$

and $\delta F/\delta f$ is defined as

$$F(f + \delta f) - F(f) = \int \frac{\delta F}{\delta f}\delta f \,\mathrm{d}(x,v) + \mathcal{O}(\delta f^2).$$

Vlasov–Poisson equation

We have

$$H_f(f+\delta f) - H_f(f) = \int \frac{1}{2} v^2 \delta f \,\mathrm{d}(x,v) \qquad \Rightarrow \qquad \frac{\delta H_f}{\delta f} = \frac{1}{2} v^2.$$

Since

$$\begin{aligned} H_E(f+\delta f) - H_E(f)\,\mathrm{d}x &= \frac{1}{2}\int \left(E(f)+E(\delta f)\right)^2 \,\mathrm{d}x - \frac{1}{2}\int E(f)^2\,\mathrm{d}x \\ &= \int E(f)\cdot E(\delta f)\,\mathrm{d}x + \mathcal{O}(\delta f^2). \end{aligned}$$

and $E = -\nabla_x \phi$ we get

$$\int \nabla_x \phi(f)\cdot \nabla_x \phi(\delta f)\,\mathrm{d}x = -\int \phi(f)\Delta_x \phi(\delta f)\,\mathrm{d}x = \int \phi(f)\rho(\delta f)\,\mathrm{d}x.$$

Thus, as $\int \phi(f)\,\mathrm{d}x = 0$ we finally have

$$\int \phi(f)\rho(\delta f)\,\mathrm{d}x = -\int \phi(f)\delta f \,\mathrm{d}(x,v) \qquad \Rightarrow \qquad \frac{\delta H_E}{\delta f} = -\phi.$$

Vlasov–Poisson equation

We can write

$$f(t,x,v)+\delta f(t,x,v)-f(t,x,v)=\delta f(t,x,v)=\int \delta(y-x)\delta(w-v)\delta f(t,y,w)\,\mathrm{d}(y,w)$$

and thus

$$\frac{\delta f(t,x,v)}{\delta f}=\delta(\cdot-x)\delta(\cdot-v).$$

Finally, we get the **Vlasov–Poisson equation** by considering

$$\begin{aligned}\partial_t f(t,x,v)&=[f(t,x,v),H_f+H_E]=\int \frac{\delta f(t,x,v)}{\delta f}\left\{\frac{\delta(H_f+H_E)}{\delta f},f\right\}\mathrm{d}(y,w)\\&=\int \delta(y-x)\delta(w-v)\left(-w\cdot\nabla_y-\nabla_y\phi(y)\cdot\nabla_w\right)f(t,y,w)\,\mathrm{d}(y,w)\\&=-v\cdot\nabla_x f+E\cdot\nabla_v f.\end{aligned}$$

Casimir invariants

Any quantity C for which $[C, H] = 0$ is an invariant.

- A **Casimir invariant** satisfies $[C, G] = 0$ for arbitrary G (stronger).

We have an **infinite number of Casimir invariants** as for any

$$C(f) = \int \psi(f) \, \mathrm{d}(x, v)$$

we get

$$\frac{\delta C}{\delta f} = \psi'(f) \quad \text{and} \quad \{\psi', f\} = \nabla_x \psi' \cdot \nabla_v f - \nabla_v \psi' \cdot \nabla_x f = \psi''(\nabla_x f \cdot \nabla_v f - \nabla_v f \cdot \nabla_x f) = 0$$

which implies $[C, G] = -[G, C] = 0$, independent of the choice of G.

Examples

- $\|f\|_2$ (in fact, any L^p norm)
- Entropy $-\int_\Omega f \log f \, \mathrm{d}(x, v)$.

Splitting

Theorem

The splitting preserves any invariant of the form

$$C(f) = \int \psi(f)\,\mathrm{d}(x, v).$$

Proof.

In the first step we have

$$\int \psi(f(x - v\Delta t, v))\,\mathrm{d}(x, v) = \int \psi(f(y, v))\,\mathrm{d}(y, v).$$

In the second step we have

$$\int \psi(f(x, v + E(x)\Delta t))\,\mathrm{d}(x, v) = \int \psi(f(x, w))\,\mathrm{d}(x, w).$$

□

Conservative splitting

More generally, **splitting schemes preserves all invariants that are preserved by the subflows**.

- Many examples of physical interest such as **mass, momentum, entropy, L^2 norm, etc.**.
- Has to be combined with an appropriate space (and possibly time) discretization in order to obtain a conservative implementation.

Proof: we apply the subflows in sequence. Since each subflow preserves the invariant the same is true for the final result.

Hamiltonian splitting

A Hamiltonian splitting is a splitting scheme that uses an additive splitting of the Hamiltonian. In our case

First subflow:

$$\partial_t f = [f, H_f] = -v \cdot \nabla_x f.$$

Second subflow:

$$\partial_t f = [f, H_E] = E \cdot \nabla_v f.$$

Thus, **the splitting by Cheng & Knorr is Hamiltonian.**

- Such methods have good long time behavior for energy (even though they do not conserve the energy exactly).

E. Hairer, G. Wanner, C. Lubich. Geometric Numerical Integration, Springer 2006.

Hamiltonian splitting for the Vlasov–Maxwell equations

Vlasov–Maxwell equations

Vlasov equation with Lorentz force

$$\partial_t f + v \cdot \nabla_x f - (E + v \times B) \cdot \nabla_v f = 0$$

coupled to **Maxwell's equations**

$$\partial_t E = c^2 \nabla_x \times B - j, \qquad \partial_t B = -\nabla_x \times E,$$

where $j = \int v f \, \mathrm{d}v$.

There are also **constraints** (automatically satisfied in the continuous case)

$$\nabla \cdot B = 0, \qquad \nabla \cdot E = \rho.$$

Hamiltonian structure

Research efforts were focused on two-term splittings.

Hamiltonian of the Vlasov–Maxwell system

$$\begin{aligned}\mathcal{H} &:= \frac{1}{2}\int E^2\,\mathrm{d}x + \frac{c^2}{2}\int B^2\,\mathrm{d}x + \frac{1}{2}\int v^2 f\,\mathrm{d}(x,v)\\ &=: H_E + H_B + H_f.\end{aligned}$$

Evolution of K

$$\partial_t K = [K, H] = [K, H_E] + [K, H_B] + [K, H_f]$$

with a highly non-canonical Poisson bracket.

Idea: Split into three terms corresponding to H_E, H_B, and H_f (Hamiltonian splitting)

N. Crouseilles, L.E., E. Faou, J. Comput. Phys. (2015).

Hamiltonian splitting

Equations for H_E

$$\partial_t f - E(x) \cdot \nabla_v f = 0, \quad \partial_t E = 0, \quad \partial_t B = -\nabla_x \times E$$

with solution

$$\begin{aligned} f(t,x,v) &= f^0(x, v + tE^0(x)), \\ E(t,x) &= E^0(x), \\ B(t,x) &= B^0(x) - t\,\nabla_x \times E^0(x). \end{aligned}$$

Equations for H_f

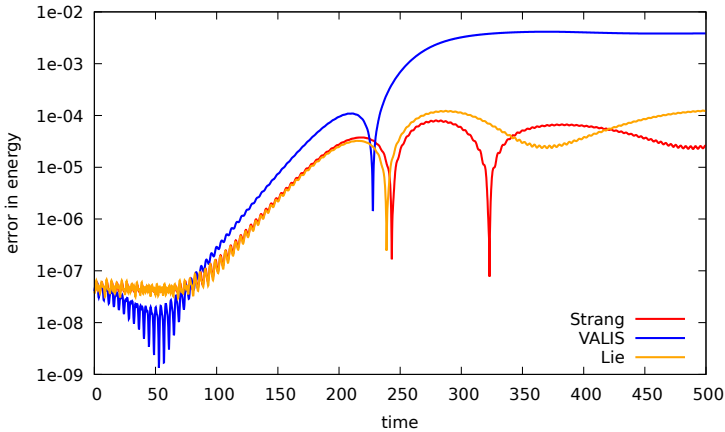
$$\partial_t f + v \cdot \nabla_x f = 0, \quad \partial_t E = -\int v f(t,x,v)\,\mathrm{d}v, \quad \partial_t B = 0.$$

Equations for H_B

$$\partial_t f + (v \times B(x)) \cdot \nabla_v f = 0, \quad \partial_t E = c^2 \nabla_x \times B, \quad \partial_t B = 0.$$

Weibel instability

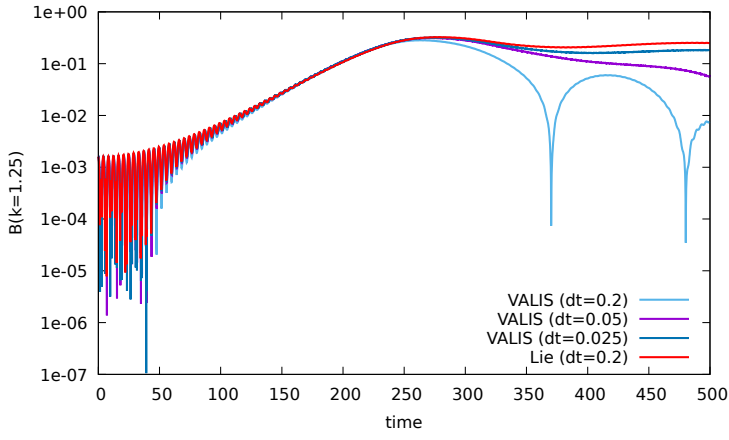
Weibel instability: temperature anisotropy makes the equilibrium unstable.



Energy conservation independent of the order of the method.

Weibel instability

Improved conservation often leads to improved qualitative properties of the solution.



Extension to higher order

Composition methods

Say $f^{n+1} = S_{\Delta t}(f^n)$ is a splitting scheme of even order p. Then

$$f^{n+1} = S_{\gamma_3 \Delta t} \circ S_{\gamma_2 \Delta t} \circ S_{\gamma_1 \Delta t}(f^n)$$

with

$$\gamma_1 = \gamma_3 = \frac{1}{2 - 2^{1/(p+1)}}, \qquad \gamma_2 = -\frac{2^{1/(p+1)}}{2 - 2^{1/(p+1)}}$$

is a method of order $p + 2$.

If we apply this to Strang splitting we get the **fourth order triple jump scheme.**

Note that we have to take **negative steps in time**.

- This is generally the case for splitting methods with order higher than two.
- For hyperbolic problems usually not an issue.

The main downside is the relatively **large number of subflows**.

Vlasov–Poisson equation

We can **use the structure of the equation to obtain more efficient methods**.

For example, for the **Vlasov–Poisson** equations we have

$$[[[H_f, H_E], H_E], H_E] = 0$$

which can be used to derive simpler order conditions.

- In the analysis of splitting schemes commutators appear in the error estimates.

6th order methods with **9 subflows** can be constructed.

- Compare with 18 subflows for repeated triple jump.
- Compare with 23 subflows for the method of Blanes & Moan.

F. Casas, N. Crouseilles, E. Faou, M. Mehrenberger. Numer. Math. 135:769–801, 2017.
S. Blanes, P.C. Moan. J. Comput. Appl. Math. 142:2, 2022.

Literature

Literature

[C.Z. Cheng, G. Knorr. J. Comput. Phys. 22:3, 1976]

- The seminal splitting scheme.

[F. Filbet, E. Sonnendrücker. Comput. Phys. Commun. 150(3), 2003]

- Comparison for many different Eulerian and semi-Lagrangian Vlasov solvers.

[N. Besse, M. Mehrenberger. Math. Comp. 77, 2008]

- Convergence analysis for a range of semi-Lagrangian schemes (including those based on cubic splines).

[L. E., A. Ostermann. SIAM J. Numer. Anal. 52:2, 2014]

- Convergence analysis for the semi-Lagrangian discontinuous Galerkin method.

Literature

[L. E., A. Ostermann. J. Numer. Anal. 52:1, 2014]

- Abstract convergence analysis for the splitting by Cheng & Knorr.

[N. Crouseilles, M. Mehrenberger, E. Sonnendrücker. J. Comput. Phys. 229:6, 2010]

- Conservative semi-Lagrangian schemes for non-constant advection in 1D.

[L.E. J. Comput. Phys. 376, 2019]

- How does conservation affect the performance of semi-Lagrangian schemes based on splines and dG.

[E. Hairer, G. Wanner, C. Lubich. Springer, 2006]

- Book about many aspects of conservative methods in the ODE context.

Literature

[P.J. Morrison. Phys. Lett. A 80, 1980]

- Hamiltonian structure for the Vlasov–Maxwell equations.

[J.E. Marsden, A. Weinstein. Phys. D.: Nonlinear Phenom. 4:3, 1980]

- Hamiltonian structure for the Vlasov–Maxwell equations that satisfies the Jacobi identity.

[N. Crouseilles, L.E., E. Faou. J. Comput. Phys. 283, 2015.]

- Hamiltonian splitting for the Vlasov–Maxwell equations.

[F. Casas, N. Crouseilles, E. Faou, M. Mehrenberger. Numer. Math. 135, 2017]

- High-order Hamiltonian splitting for Vlasov–Poisson.