Integration

Riemann integral

$f : [a, b] \to \mathbb{R}, \quad f$ bounded
$a = x_0 \le x_1 \le \cdots \le x_n = b \qquad$ Partition $P \in \mathfrak{P}$

$$\begin{aligned}
\Delta x_i &= x_i - x_{i-1} \\
M_i &= \sup_{x \in [x_{i-1}, x_i]} f(x) \\
m_i &= \inf_{x \in [x_{i-1}, x_i]} f(x) \\
U(P, f) &= \sum_{i=1}^{n} M_i \Delta x_i \\
L(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i \\
\overline{\int_b^a} f\, dx &= \inf_{P \in \mathfrak{P}} U(P, f) \\
\underline{\int_b^a} f\, dx &= \sup_{P \in \mathfrak{P}} L(P, f)
\end{aligned}$$

$$[\overline{\int_b^a} f\, dx = \underline{\int_b^a} f\, dx] \implies [\int_a^b f\, dx = \overline{\int_b^a} f\, dx]$$

We call the function Riemann integrable ($f \in \mathfrak{R}$).

Riemann Stieltjes integral

We proceed as we would with the Riemann integral but define

$g : [a, b] \to \mathbb{R}, \quad g$ monotonically increasing
$\Delta x_i = g(x_i) - g(x_{i-1})$

If the integral exists we write

$\int_a^b f\, dg$ or $\int_a^b f(x)\, dg(x)$ with $f \in \mathfrak{L}(g)$

The Riemann integral is a special case of the Riemann Stieltjes integral.

Improper (Riemann) integral

$$\begin{aligned}
\int_a^{\infty} f\, dx &= \lim_{b \to \infty} \int_a^b f\, dx \\
\int_{-\infty}^{b} f\, dx &= \lim_{a \to -\infty} \int_a^b f\, dx \\
\int_{-\infty}^{\infty} f\, dx &= \lim_{a \to -\infty} \int_a^c f\, dx + \lim_{b \to \infty} \int_c^b f\, dx \\
\int_a^b f\, dx &= \lim_{d \to c-} \int_a^d f\, dx + \lim_{e \to c+} \int_e^b f\, dx
\end{aligned}$$

Integrability

$f \in \mathfrak{L}(h) \iff$
$\forall \epsilon > 0 \exists P \in \mathfrak{P} \ : \ U(P, f, h) - L(P, f, a) < \epsilon$

$f \in \mathcal{C}^0 \implies f \in \mathfrak{L}(h)$

f monotone $\wedge \ g \in \mathcal{C}^0 \implies \quad f \in \mathfrak{L}(h)$

If f is bounded and continuous for almost all points, and g is continuous at every discontinuity of f then $f \in \mathfrak{L}(g)$.

$g \in \mathcal{C}^1 \implies$
$\int_a^b f\, dg = \int_a^b f \cdot g'\, dx$

Fundamental theorem of calculus

$f \in \mathcal{C}^0([a, b]) \implies$

$$\begin{aligned}
F(x) &= \int_a^x f(t)\, dt \\
F'(x) &= f(x)
\end{aligned}$$

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Elementary functions

$f(x)$	$\to$	$\int f(x)\, dx$	
c	$\to$	$c \cdot x$	
cx	$\to$	$c \cdot \frac{x^2}{2}$	
x^c	$\to$	$\frac{x^{c+1}}{c+1}$	$c \neq -1$
$\frac{1}{x}$	$\to$	$\log x$	
c^x	$\to$	$\frac{c^x}{\log c}$	
$\log_c x$	$\to$	$\frac{x \log x - x}{\log c}$	$c > 0, c \neq 1$
$\frac{1}{x^c}$	$\to$	$\frac{-x}{(c-1)x^c}$	$x \neq 0, \ c \neq 1$
$\sqrt{x}$	$\to$	$\frac{2}{3} x^{3/2}$	$x \geq 0$
e^x	$\to$	e^x	
$\ln x$	$\to$	$x \cdot \log x - x$	$x > 0$

Hyperbolic functions

$f(x)$	$\to$	$\frac{d}{dx} f(x)$
$\sinh x$	$\to$	$\cosh x$
$\cosh x$	$\to$	$\sinh x$
$\tanh x$	$\to$	$\ln(\cosh x)$
$\coth x$	$\to$	$\ln(\sinh x)$
$\mathrm{arsinh} x$	$\to$	$x \cdot \mathrm{arsinh} x - \sqrt{1 + x^2}$
$\mathrm{arcosh} x$	$\to$	$x \cdot \mathrm{arcosh} x - \sqrt{x - 1}\sqrt{x + 1}$
$\mathrm{artanh} x$	$\to$	$x \cdot \mathrm{artanh} x + \frac{1}{2} \ln(1 - x^2)$
$\mathrm{arcoth} x$	$\to$	$x \cdot \mathrm{arcoth} x + \frac{1}{2} \ln(x^2 - 1)$
$\mathrm{csch} x = \frac{1}{\sinh x}$	$\to$	$\ln(\tanh(\frac{x}{2}))$
$\mathrm{sech} x = \frac{1}{\cosh x}$	$\to$	$\arctan(\sinh(x))$

Trigonometric functions

$f(x)$	$\to$	$\frac{d}{dx} f(x)$
$\sin x$	$\to$	$-\cos x$
$\cos x$	$\to$	$\sin x$
$\tan x$	$\to$	$-\ln(\cos x)$
$\cot x$	$\to$	$\ln(\sin x)$
$\arcsin x$	$\to$	$x \cdot \arcsin x + \sqrt{1 - x^2}$
$\arccos x$	$\to$	$x \cdot \arccos x - \sqrt{1 - x^2}$
$\arctan x$	$\to$	$x \cdot \arctan x - \frac{1}{2} \ln(1 + x^2)$
$\mathrm{arccot} x$	$\to$	$\frac{\pi}{2} x - x \cdot \arctan x + \frac{1}{2} \ln(1 + x^2)$
$\csc x = \frac{1}{\sin x}$	$\to$	$-\ln\left(\frac{1 + \cos x}{\sin x}\right)$
$\sec x = \frac{1}{\cos x}$	$\to$	$\ln\left(\frac{1 + \sin x}{\cos x}\right)$

Linearity of the integral

$f, g \in \mathfrak{L}(h), \ c \in \mathbb{R}$

$$\begin{aligned}
\int_a^b f + g\, dh &= \int_a^b f\, dh + \int_a^b g\, dh \\
\int_a^b c \cdot f\, dh &= c \cdot \int_a^b f\, dh \\
\int_a^c f\, dh + \int_c^b f\, dh &= \int_a^b f\, dh \\
\int_a^b f\, d(h_1 + h_2) &= \int_a^b f\, dh_1 + \int_a^b f\, dh_2 \\
\int_a^b f\, d(ch) &= c \cdot \int_a^b f\, dh
\end{aligned}$$

Integration by parts

$$\begin{aligned}
\int f \cdot g'\, dx &= f \cdot g - \int f' \cdot g\, dx \\
\int f \cdot g\, dx &= f \cdot G - \int f' \cdot G\, dx
\end{aligned}$$

Choose f as the function that comes first in **ILATE** (inverse trigonometric, logarithmic, algebraic, trigonometric, exponential).

Integration by substitution

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(y)\, dy \qquad \begin{aligned} y &= g(x) \\ dy &= g'(x)\, dx \end{aligned}$$

Integration of rational functions

Every rational function ($\mathcal{R}(x) = P(x)/Q(x)$) is integrable (where the function is bounded).

(1) $\deg P \geq \deg Q \implies$ polynomial division
(2) $\deg P < \deg Q \implies$ partial fraction decomposition

If real partial fraction decomposition is used then by completing the square and the substitution with inverse trigonometric/hyperbolic function the integral can be computed.

If complex partial fraction decomposition is used the resulting fractions are trivial.

$$\ln(a + b\imath) = \frac{1}{2} \ln(a^2 + b^2) + \imath \arctan\left(\frac{b}{a}\right), \ a > 0$$

Integration by reduction to a rational function

(a) $\int \mathcal{R}(x, \sqrt[n]{ax + b})\, dx = \int \mathcal{R}(\frac{y^n - b}{a}, y) \frac{n}{a} y^{n-1} dy$
$x = \frac{1}{a}(y^n - b) \quad dx = \frac{n}{a} y^{n-1} dy$

(b) $\int \mathcal{R}(x, \sqrt{ax^2 + bx + c})\, dx$
completion of the square and linear substitution then (c)

(c) $\int \mathcal{R}(y, \sqrt{y^2 \pm 1})\, dy \ \vee \ \int \mathcal{R}(y, \sqrt{1 - y^2})\, dy$
substitute with *sinh/cosh/cos*

(d) $\int \mathcal{R}(e^{ax})\, dx = \int \mathcal{R}(y) \frac{1}{ay}\, dy$
$y = e^{ax} \quad dx = \frac{1}{ay} dy$

(e) $\int \mathcal{R}(\sin x, \cos x)\, dx = \int \mathcal{R}(\frac{1 - y^2}{1 + y^2}, \frac{2y}{1 + y^2}) \frac{2}{1 + y^2}\, dy$
$y = \tan(\frac{x}{2})$

Line/path integral of a scalar field

$f : U \subset \mathbb{R}^n \to \mathbb{R}, \quad \boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ is a parameterization of C
$\int_C f\, ds = \int_a^b f(\boldsymbol{\gamma}(t)) \cdot ||\boldsymbol{\gamma}'||\, dt$

$\int_C c \cdot (f + g)\, ds = \left[c \cdot \int_C f\, ds + \int_C g\, ds\right]$ — Linearity
$\int_y f\, ds = \int_{\gamma \circ \varphi} f\, ds$ — φ is any change in parametrization

Line/path integral of a vector field

$\boldsymbol{f} : U \subset \mathbb{R}^n \to \mathbb{R}^n, \quad \boldsymbol{\gamma} : [a, b] \to \mathbb{R}^n$ is a parameterization of C
$\int_C \langle \boldsymbol{f}, d\boldsymbol{x} \rangle\, ds = \int_a^b \langle \boldsymbol{f}(\boldsymbol{\gamma}(t)), \boldsymbol{\gamma}'(t) \rangle\, dt$

$\int_\gamma \langle c \cdot (\boldsymbol{f} + \boldsymbol{g}), d\boldsymbol{x} \rangle = c \cdot \left[\int_\gamma \langle \boldsymbol{f}, d\boldsymbol{x} \rangle + \int_\gamma \langle \boldsymbol{g}, d\boldsymbol{x} \rangle\right]$ — Linearity
$\int_\gamma \langle \boldsymbol{f}, d\boldsymbol{x} \rangle = \int_{\gamma \circ \varphi} \langle \boldsymbol{f}, d\boldsymbol{x} \rangle$ — φ is any increasing change in parametrization

$\int_\gamma \langle \boldsymbol{f}, d\boldsymbol{x} \rangle = -\int_{\gamma \circ \varphi} \langle \boldsymbol{f}, d\boldsymbol{x} \rangle$ — φ is any decreasing change in parametrization

If $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$ we write $\oint_C \langle \boldsymbol{f}, d\boldsymbol{x} \rangle$ and call C a closed curve.
$\int_C \langle \boldsymbol{f}, d\boldsymbol{x} \rangle = \int_C f_1 dx_1 \ + \ \ldots \ + \ f_n dx_n$

Potential field

$\boldsymbol{v}: D \subset \mathbb{R}^n \to \mathbb{R}^n,\ D$ region

If $\boldsymbol{v}(\boldsymbol{x}) = \nabla\varphi(\boldsymbol{x})$ we call $\boldsymbol{v}$ a **potential field**/scalar field/gradient field/conservative field.

We call φ an **antiderivative** of $\boldsymbol{v}$ and $-\varphi$ the **potential** of $\boldsymbol{v}$.

$$\begin{aligned}\int_\gamma \langle \boldsymbol{v}, d\boldsymbol{x}\rangle &= \varphi(\gamma(b)) - \varphi(\gamma(a))\\ \oint_\gamma \langle \boldsymbol{v}, d\boldsymbol{x}\rangle &= 0\end{aligned}$$

Vector fields that are not potential fields can often be decomposed into a potential field and a simpler non-potential vector field.

Potential field test

D star domain, integrability condition
$\Longrightarrow \boldsymbol{f}$ potential field
$\frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2}$ integrability condition (2D)
$\nabla \times \boldsymbol{f} = \boldsymbol{0}$ integrability condition (3D)

Antiderivative (method 1)

$\varphi(\boldsymbol{x}) = \int_C \langle \boldsymbol{v}, d\boldsymbol{x}\rangle$
where the endpoint of C is $\boldsymbol{x}$ and the initial point is fixed.

Antiderivative (method 2)

$$\boldsymbol{v} = \begin{bmatrix} \varphi_{x_1} \\ \vdots \\ \varphi_{x_n}\end{bmatrix}$$

We determine φ (by integration) up to a a function $C(x_2,\dots,x_n)$; then, with v_2 we determine φ up to a function of $C(x_3,\dots,x_n)$and so on.

This method leads to a contradiction if $\boldsymbol{v}$ is not a potential field.

Multiple integral

$\int \dots \int_D f(x_1,\dots,x_n)\ dx_1 \dots dx_n$ or $\int_D f\ d\boldsymbol{x}$

The multiple integral is linear.

$D \subset R = [a,b]\times[c,d]$
$\int_D f\ d(x_1,\dots,x_n) = \int_R \mathbf{1}_D \cdot f\ d(x_1,\dots,x_n)$

Fubini's theorem

$$\begin{aligned}\int_{A\times B} f\ d(x,y) &= \int_A [\int_B f\ dy]\ dx = \int_B [\int_A f\ dx]\ dy\\ \int_{[a,b]\times[c,d]} f\ d(x,y) &= \int_a^b \int_c^d [f\ dy]\ dx = \int_c^d \int_a^b [f\ dx]\ dy\end{aligned}$$

Computation: $\mathbb{R}^2$

$f(x,y)\cdot \mathbf{1}_D(x,y) \in \mathfrak{L}$

$D = \{(x,y) | a \le x \le b, l(x) \le y \le u(x)\}$
$\Longrightarrow \int_D f\ d(x,y) = \int_a^b \left(\int_{l(x)}^{u(x)} f\ dy\right) dx$

$D = \{(x,y) | l(y) \le x \le r(y), c \le y \le d\}$
$\Longrightarrow \int_D f\ d(x,y) = \int_c^d \left(\int_{l(y)}^{r(y)} f\ dx\right) dy$

Computation: $\mathbb{R}^3$

$D = \{(x,y,z) | a \le x \le b, u(x) \le y \le v(x), g(x,y) \le z \le h(x,y)\}$
$\Longrightarrow \int_D f\ d(x,y,z) = \int_a^b \left(\int_{u(x)}^{v(x)} \left(\int_{g(x,y)}^{h(x,y)} f\ dz\right) dy\right) dx$

Transformation

$B, D \subset \mathbb{R}^n$, bounded, open
$\boldsymbol{F}: D \to B$, diffeomorphism
$f: B \to \mathbb{R}$, bounded
$\int_{\boldsymbol{F}(D)} f\ d\boldsymbol{x} = \int_D f(\boldsymbol{F}(\boldsymbol{u})) |\det \boldsymbol{F}'|\ d\boldsymbol{u}$

Improper integrals

$D \subset \mathbb{R}^n$, open
$f: D \to \mathbb{R}^n,\ f \in \mathcal{C}^0$
$\overline{A_j} \subset A_{j+1} \ \wedge\ \bigcup_{j=1}^{\infty} A_j = D$
$\int_D f\ d\boldsymbol{x} = \lim_{j\to\infty} \int_{A_j} f\ d\boldsymbol{x}$
If the integral exists and is equal for all A

Divergence theorem/Gauss's theorem

$B \subset \mathbb{R}^n$, compact, ∂B piecewise smooth boundary
$\boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^n,\ \boldsymbol{f} \in \mathcal{C}^1(B) \cap \mathcal{C}^0(\partial B)$
$\Longrightarrow \int_B \operatorname{div}\boldsymbol{f}\ d(x,y) = \oint_{\partial V} \langle \boldsymbol{f}, d\boldsymbol{n}\rangle$

Green's theorem

$B \in \mathbb{R}^2$, positively oriented, ∂B
∂B piecewise smooth, simple closed curve in $\mathbb{R}^2$
$\Longrightarrow \int_B \operatorname{curl}\boldsymbol{f}\ d(x,y) = \oint_{\partial B} \langle \boldsymbol{f}, d\boldsymbol{x}\rangle$

Green's theorem is the 2D special case of Stokes' theorem.

Kelvin-Stokes theorem/Stokes' theorem

$\oint_{\partial S} \langle \boldsymbol{f}, d\boldsymbol{x}\rangle = \int_S \langle \operatorname{curl}\boldsymbol{f}, d\boldsymbol{S}\rangle$

Surface (Parametrization)

$D \subset \mathbb{R}^2,\ D$ open, connected

$$\boldsymbol{\Phi}: D \to \mathbb{R}^3 : (u,v) \mapsto \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v)\end{bmatrix}$$

Surface integral of a potential field

$S = \boldsymbol{\Phi}(B) \subset \mathbb{R}^3,\ \boldsymbol{\Phi}$ regular
$f: S \to \mathbb{R},\quad f \in \mathcal{C}^0$
$\int_S f\ dS = \int_B f(\boldsymbol{\Phi}(u,v)) \cdot |\frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}|\ d(u,v)$

Surface integral of a vector field

$S = \boldsymbol{\Phi}(B) \subset \mathbb{R}^3,\ \boldsymbol{\Phi}$ regular
$\boldsymbol{f}: S \to \mathbb{R}^3,\quad \boldsymbol{f} \in \mathcal{C}^0$
$\int_S \langle \boldsymbol{f}, d\boldsymbol{S}\rangle = \int_B \langle \boldsymbol{f}(\boldsymbol{\Phi}(u,v)), \frac{\partial \boldsymbol{\Phi}}{\partial u} \times \frac{\partial \boldsymbol{\Phi}}{\partial v}\rangle\ d(u,v)$

Area/Volume

The integral is the (signed) area under a function. The double integral is the (signed) volume under a function and $\int_D 1\ d(x,y)$ is the area of the set D. The triple integral $\int_D 1\ d(x,y,z)$ is the volume of the set D.

More applications can be found in the sciences (mass from density, center of mass, moment of inertia, ...).

Arc length

$$\begin{aligned} s &= \int_a^b ||\dot{\boldsymbol{x}}(t)||\ dt\\ s &= \int_a^b \sqrt{1 + f'(x)}\ dx\\ ds &= ||\dot{\boldsymbol{x}}(t)||\ dt\end{aligned}$$

Solid of revolution

$$\begin{aligned} V &= \pi \int_a^b f(x)^2\ dx && x \text{ is aor}\\ V &= 2\pi \int_a^b x \cdot f(x)\ dx && y \text{ is aor}\end{aligned}$$

Surface of revolution (of a function)

$$\begin{aligned} M &= 2\pi \int_a^b f(x)\ ds && x \text{ is aor}\\ M &= 2\pi \int_a^b f(x) \cdot \sqrt{1 + f'(x)^2}\ dx && x \text{ is aor}\\ M &= 2\pi \int_a^b f^{-1}(y) \cdot \sqrt{1 + ((f^{-1}(y))')^2}\ dy && y \text{ is aor}\end{aligned}$$

Surface of revolution (of a curve)

$$\begin{aligned} M &= 2\pi \int_a^b x_2(t)\ ds && x \text{ is aor}\\ M &= 2\pi \int_a^b x_1(t)\ ds && y \text{ is aor}\end{aligned}$$

aor stands for axis of revolution.