Curves

Definition

$\gamma : [a,b] \to \mathbb{R}^n$ continuous — path
$\gamma([a,b])$ — curve

A path is **closed** if $\gamma(a) = \gamma(b)$.

A path is **simple** if γ is injective.

A **Jordan path** is a simple closed curve (in german only simple is required for a path to be a Jordan path).

A path is **smooth** if $\gamma \in \mathcal{C}^1([a,b])$ and $\gamma' \neq \mathbf{0}$.

A path is **piecewise smooth** if it can be separated into a finite number of smooth paths.

A curve has any of the properties mentioned above if there exists a path for which the property holds true. We then call the path a **parametrization** of the curve.

The parametrization of a curve is (in general) not unique.

Special curves

Some curves can be expressed as

$$f_1(x_1,\dots,x_n),\dots,f_{n-1}(x_1,\dots,x_n) = 0 \quad \text{implicite form}$$

A curve is called algebraic iff it is expressible in implicite form were f are polynomial functions.

Jordan-curve theorem

The complement of the image of a simple-closed curve in $\mathbb{R}^2$ consists of two distinct components. The interior is bounded while the exterior is unbounded. The curve is the boundary of both components.

Parametrization of well-known curves

Line segment from $a \in \mathbb{R}^n$ to $b \in \mathbb{R}^n$ (simple)
$$\gamma(t) = a + t(b-a) = (1-t)a + tb \qquad 0 \le t \le 1$$

Circle with radius r in $\mathbb{R}^2$ and $x^2 + y^2 = r^2$ (simple and closed)
$$\gamma(t) = \begin{bmatrix} r\cdot\cos t \\ r\cdot\sin t \end{bmatrix} \qquad 0 \le t \le 2\pi$$

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$ (simple and closed)
$$\gamma(t) = \begin{bmatrix} a\cdot\cos t \\ b\cdot\sin t \end{bmatrix} \qquad 0 \le t \le 2\pi$$

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (simple)
$$\gamma(t) = \begin{bmatrix} \pm a\cdot\cosh t \\ b\cdot\sinh t \end{bmatrix} \qquad t \in \mathbb{R}$$

Parametrization of a curve in explicite form (simple, not closed)

$$\gamma(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_{n-1}(t) \\ t \end{bmatrix}$$

Parametrization of a curve in (explicite) polar coordinates $r(\varphi)$

$$\gamma(t) = \begin{bmatrix} r(\varphi)\cdot\cos\varphi \\ r(\varphi)\cdot\sin\varphi \end{bmatrix}$$

Equivalence

$$\gamma : I \to \mathbb{R}^n \sim \beta : J \to \mathbb{R}^n \iff$$
$$\exists h : J \to I, \ \text{bijective, continuous, increasing}$$
$$\forall t \in J: \ \beta(t) = \gamma(h(t))$$

Curves which are equivalent are said to have the same orientation.

If h is decreasing the orientation is reversed.

For two simple curves γ and β the bijection h is unique; thus, up to equivalence and orientation the parametrization of a simple curve is unique.

Arc length

A curve is called rectifiable if its arc length L is finite.

If $\gamma : [a,b] \to \mathbb{R}^n$ is differentiable then
$$s(t) = \int_a^t ||\gamma'(\tau)||_2 \, d\tau$$
$$L(\gamma) = \int_a^b ||\gamma'(t)||_2 \, dt$$

Every continuous differentiable curve is rectifiable.

Every piecewise continuous differentiable curve is rectifiable.

Arc length (special cases)

$f \in \mathcal{C}^1([a,b])$
$$\Longrightarrow L = \int_a^b \sqrt{1 + f'(x)} \, dx$$

$r(\varphi) \in \mathcal{C}^1([\alpha,\beta])$
r is piecewise simple
$$L = \int_\alpha^\beta \sqrt{r^2(\varphi) + (r')^2(\varphi)} \, \varphi$$

Every function in polar coordinates with $\beta - \alpha < 2\pi$ is a simple curve.

Natural parameter

A rectifiable piecewise simple curve has a parametrization with the arc-length as its parameter ($\gamma(s)$). Then $||\gamma'(s)||_2 = 1$. We call this parametetrization **natural**.

There exist exactly two different natural parametrizations (up to the choice of a starting point of a closed curve); these two have a different orientation.

Polar coordinate system

$$F(r,\varphi) = \begin{bmatrix} r\cdot\cos\varphi \\ r\cdot\sin\varphi \end{bmatrix}$$
$$F^{-1}(x,y) = \begin{bmatrix} \sqrt{x^2+y^2} \\ \arctan\frac{y}{x} \end{bmatrix}$$
$y \in \mathbb{R}$, $x \in (0,\infty)$, $r \in (0,\infty)$, $\varphi \in (\frac{-\pi}{2}, \frac{\pi}{2})$

Velocity/Acceleration

$\gamma'(t)$	velocity
$\gamma''(t)$	acceleration
$\lvert\lvert y'(s)\rvert\rvert_2 = 1$	arc length s

Curvature in $\mathbb{R}^2$

$$T(t) = \frac{\gamma'}{||\gamma'||_2} = \gamma'(s) \quad \text{unit tangent-vector}$$
$$N(t) = \frac{(\gamma')^\perp}{||\gamma'||_2} \quad \text{unit normal-vector}$$

$\kappa(s) = \lvert\lvert\gamma''(s)\rvert\rvert$	curvature
$r(s) = 1/\kappa(s)$	radius of curvature
$\kappa = \frac{y''}{(1+(y')^2)^{3/2}}$	$y(x)$
$\kappa = \frac{x'y''-y'x''}{((x')^2+(y')^2)^{3/2}}$	$x(t)$, $y(t)$

Tangent

$$x(t) = \gamma(t_0) + t\cdot\gamma'(t_0), \quad t \in \mathbb{R}$$

Torsion in $\mathbb{R}^3$

$$T(t) = \frac{\gamma'}{||\gamma||_2}$$
$$N(t) = \frac{T'}{||T||_2}$$
$$B(t) = T \times N \quad \text{binormal-unit-vector}$$
$$\langle T, N\rangle = 0$$

$$\kappa(t) = \frac{||T(t)||_2}{||\gamma'||_2} = T'(s)$$
$$\kappa(t) = \frac{||\gamma'\times\gamma''||_2}{||\gamma'||_2^3}$$
$$\tau(t) = \frac{\det[\gamma',\gamma'',\gamma''']}{||\gamma'\times\gamma''||_2} \quad \text{torsion}$$

Frenet-Serret formulas in $\mathbb{R}^3$

$$T'(s) = \kappa N$$
$$N'(s) = -\kappa T + \tau B$$
$$B'(s) = -\tau N$$

Osculating plane

$$E(\lambda,\mu) = \gamma(t_0) + \lambda T(t_0) + \mu N(t_0)$$

A bijective, partially continuous differentiable function $F : U \to V$ ($u, v \in \mathbb{R}^n$ and open) is called **diffeomorphism** iff F^{-1} is partially continuous differentiable. We say $x = F(u)$ defines **local coordinates** $u = (u_1,\dots,u_n)$.

Spherical coordinate system

$$F(r,\varphi,\theta) = \begin{bmatrix} r\cdot\sin\theta\cos\varphi \\ r\cdot\sin\theta\sin\varphi \\ r\cdot\cos\theta \end{bmatrix}, \quad F^{-1}(x,y,z) = \begin{bmatrix} \sqrt{x^2+y^2+z^2} \\ \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \arctan\frac{y}{x} \end{bmatrix}$$
$y, z \in \mathbb{R}$, $x \in (0,\infty)$, $r \in (0,\infty)$, $\varphi \in (\frac{-\pi}{2}, \frac{\pi}{2})$, $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$

Cylindrical coordinate system

$$F(r,\varphi,\theta) = \begin{bmatrix} r\cdot\cos\varphi \\ r\cdot\sin\varphi \\ z \end{bmatrix}, \quad F^{-1}(x,y,z) = \begin{bmatrix} \sqrt{x^2+y^2} \\ \arctan\frac{y}{x} \\ z \end{bmatrix}$$
$y, z \in \mathbb{R}$, $x \in (0,\infty)$, $r \in (0,\infty)$, $\varphi \in (\frac{-\pi}{2}, \frac{\pi}{2})$